

Higher order representation stability and ordered configuration spaces of manifolds

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Using the language of twisted skew-commutative algebras, we define *secondary representation stability*, a stability pattern in the *unstable* homology of spaces that are representation stable in the sense of Church, Ellenberg, and Farb [CEF15]. We show that the rational homology of configuration spaces of ordered particles in noncompact manifolds satisfies secondary representation stability. While representation stability for the homology of configuration spaces involves stabilizing by introducing particles near the boundary, secondary representation stability involves stabilizing by introducing pairs of orbiting particles. This result can be thought of as a representation-theoretic analogue of *secondary homological stability* in the sense of Galatius, Kupers, and Randal-Williams [GKRW]. In the course of the proof we establish some additional results: we give a new characterization of the integral homology of the complex of injective words, and we give a new proof of integral representation stability for configuration spaces of noncompact manifolds, extending previous results to nonorientable manifolds. In an appendix, we use results on FI-homology to give explicit stable ranges for the integral cohomology of configuration spaces of closed manifolds.

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1 Introduction

The object of this paper is to introduce the concept of *secondary representation stability* and prove that this phenomenon is present in the homology of the ordered configuration spaces of a connected noncompact manifold. Church, Ellenberg, and Farb [CEF15] proved that, in each fixed homological degree i , these homology groups are *representation stable*: up to the action of the symmetric groups, the homology classes stabilize under the operation of adding a particle “at infinity.” In this paper, we exhibit patterns between *unstable* homology groups in different homological degrees. We show that certain sequences of unstable homology groups stabilize under the new operation of adding *pairs* of particles orbiting each other “at infinity.” We formalize this secondary representation stability phenomenon using the language of twisted skew-commutative algebras.

1.1 Stability for configuration spaces

For a manifold M , let $F_k(M) = \{(m_1, \dots, m_k) \mid m_i \in M, m_i \neq m_j \text{ if } i \neq j\} \subseteq M^k$ be the configuration space of k distinct ordered particles in M . The symmetric group \mathfrak{S}_k acts on $F_k(M)$ by permuting the coordinates, and so induces a $\mathbb{Z}[\mathfrak{S}_k]$ -module structure on the homology groups $H_i(F_k(M))$. Although these homology groups do not exhibit classical homological stability as k increases, Church, Ellenberg, and Farb [Chu12, CEF15] showed that they do stabilize in a certain sense as symmetric group representations. To make this statement of *representation stability* precise, we recall the definition of the stabilization map.

Assume throughout that M is a connected noncompact n -manifold with $n \geq 2$. Since M is not compact, there is an embedding $e : M \sqcup \mathbb{R}^n \hookrightarrow M$ such that $e|_M$ is isotopic to the identity, as in Figure 1. Such an embedding exists, for example, by Kupers–Miller [KM15a, Lemma 2.4]. Using this

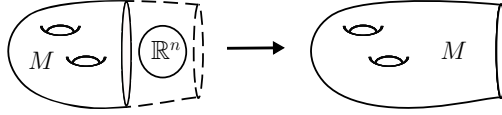


Figure 1: The embedding $e : M \sqcup \mathbb{R}^n \hookrightarrow M$.

embedding, we construct a map

$$t : F_{k-1}(M) \rightarrow F_k(M)$$

which maps a configuration in M to its image in $e(M)$, and then adds a particle labeled by k in $e(\mathbb{R}^n)$. This map is illustrated in Figure 2.

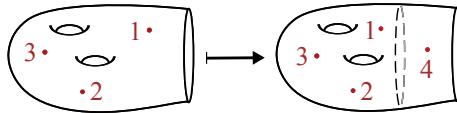


Figure 2: The stabilization map $t : F_3(M) \rightarrow F_4(M)$.

Building on the work of Church [Chu12], Church, Ellenberg, and Farb [CEF15] proved the following stability theorem for the homology groups of ordered configuration spaces.

Theorem 1.1 ([CEF15, Theorem 6.4.3]). *Let M be a connected, orientable, noncompact n -manifold with $n \geq 2$. For $i \leq \frac{k-1}{2}$,*

$$\mathbb{Z}[\mathfrak{S}_k] \cdot t_*(H_i(F_{k-1}(M); \mathbb{Z})) = H_i(F_k(M); \mathbb{Z}).$$

Under the action of \mathfrak{S}_k , the image $t_*(H_i(F_{k-1}(M); \mathbb{Z}))$ generates all of $H_i(F_k(M); \mathbb{Z})$.

In this paper, we propose a higher-order stabilization map, t' . Using the embedding e we can also construct a map $F_{k-2}(M) \times F_2(\mathbb{R}^n) \rightarrow F_k(M)$ which places two particles in $e(\mathbb{R}^n)$, labeled by

$(k-1)$ and k . This induces a map $H_a(F_{k-2}(M)) \otimes H_b(F_2(\mathbb{R}^n)) \rightarrow H_{a+b}(F_k(M))$. We then define the stabilization map

$$t' : H_{i-1}(F_{k-2}(M)) \rightarrow H_i(F_k(M))$$

by pairing a class in $H_{i-1}(F_{k-2}(M))$ with the class in $H_1(F_2(\mathbb{R}^n))$ of the particle $(k-1)$ orbiting the particle k counterclockwise, as in Figure 3. Note that this operation is symmetric in k and $(k-1)$. While the classical stabilization map t_* raises the number of particles by one and keeps homological

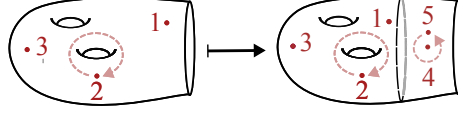


Figure 3: The secondary stabilization map $t' : H_1(F_3(M)) \rightarrow H_2(F_5(M))$.

degree constant, the map t' increases the number of particles by two and homological degree by one.

With the definition of t' , we can state the following version of our main theorem, *secondary representation stability* for the homology of configuration spaces. For this theorem we do not need to assume M is orientable, but we assume that our manifold M is finite type (that is, the homotopy type of a finite CW complex) to ensure that the homology groups of the configuration spaces are finite dimensional. Let \mathbb{N}_0 denote the set of nonnegative integers.

Theorem 1.2. *Let M be a connected noncompact finite type n -manifold with $n \geq 2$. There is a function $r : \mathbb{N}_0 \rightarrow \mathbb{N}_0$ tending to infinity such that for $i \leq \frac{k-1}{2} + r(k)$,*

$$\mathbb{Q}[\mathfrak{S}_k] \cdot \left(t_*(H_i(F_{k-1}(M); \mathbb{Q})) + t'(H_{i-1}(F_{k-2}(M); \mathbb{Q})) \right) = H_i(F_k(M); \mathbb{Q}).$$

Up to the action of \mathfrak{S}_k , the homology group $H_i(F_k(M); \mathbb{Q})$ is generated by the images of t_* and t' . In other words, Theorem 1.1 says that when the homological degree i is small enough relative the number k of particles, the group $H_i(F_k(M); \mathbb{Q})$ is spanned by classes where at least one particle is stationary near the boundary. Theorem 1.2 says that there is a larger range in which the homology group is spanned by classes where at least one particle is stationary, or two particles are orbiting each other near the boundary.

Remark 1.3. The idea to study homological degree-shifting stabilization maps originated with the work of Galatius, Kupers, and Randal-Williams [GKRW]. Their work generalizes classical homological stability, whereas we generalize representation stability. See also Hepworth [Hep16, Theorem B] for a related result on certain families of groups.

1.2 Categorical reformulation

In order to prove Theorem 1.2, and interpret it within the broader field of representation stability, we will reformulate the result in terms of finite generation of a module over a certain enriched category (or equivalently as a module over a certain *twisted skew-commutative algebra*). From this perspective, Theorem 1.2 becomes a structural algebraic result on the homology of configuration spaces. We now review elements of the theory of representation stability and FI-modules.

Church–Farb [CF13] introduced the concept of *representation stability* to generalize homological stability to sequences of spaces that are not homologically stable in the classical sense, but that admit actions by a family of groups \mathcal{G}_k . Representation stability was originally conceived for groups \mathcal{G}_k , $k \in \mathbb{N}_0$, and a coefficient ring R for which there are natural identifications between irreducible modules over $R[\mathcal{G}_k]$ and $R[\mathcal{G}_{k+1}]$. A principal example is the family of symmetric groups \mathfrak{S}_k over the rational numbers. A sequence \mathcal{V}_k of $R[\mathcal{G}_k]$ -modules with maps $\phi_k : \mathcal{V}_k \rightarrow \mathcal{V}_{k+1}$ is called *representation stable* if the multiplicities of the irreducible subrepresentations of \mathcal{V}_k are eventually constant, in a manner compatible with the maps ϕ_k . Church and Farb described a wide range of examples and applications of this phenomenon, and Church [Chu12] proved that the sequence $\{H^i(F_k(M); \mathbb{Q})\}_{k=0}^\infty$ has representation

stability as \mathfrak{S}_k -representations for any finite-type connected orientable manifold M , including closed manifolds.

This formulation of representation stability, however, does not easily generalize to situations without a complete classification of irreducible representations of the groups \mathcal{G}_k , or when the group rings $R[\mathcal{G}_k]$ are not semisimple. Church, Ellenberg, and Farb [CEF15] successfully generalized representation stability for sequences of symmetric group representations by reframing the theory in terms of finiteness properties of algebraic objects called FI-modules. In this setting, “representation stability” for sequences of symmetric group representations makes sense over arbitrary coefficient rings R , moreover, it creates a framework to study these sequences using tools from category theory and homological algebra.

FI-modules

Let \mathbf{FI} denote the category of finite sets and injective maps. An *FI-module* (over a commutative unital ring R) is a covariant functor \mathcal{V} from \mathbf{FI} to the category of R -modules. Given an FI-module \mathcal{V} , we write \mathcal{V}_S to denote the image of \mathcal{V} on a set S , or for $k \in \mathbb{N}_0$ we let \mathcal{V}_k denote the value of \mathcal{V} on the standard set $[k] := \{1, \dots, k\}$ or $[0] = \emptyset$. The endomorphisms $\text{End}([k]) \cong \mathfrak{S}_k$ induce an action of the symmetric group \mathfrak{S}_k on \mathcal{V}_k . The FI-module structure on \mathcal{V} is completely determined by these symmetric group actions and the maps $\mathcal{V}_k \rightarrow \mathcal{V}_{k+1}$ induced by the standard inclusions $[k] \subset [k+1]$.

Given an FI-module \mathcal{V} , the *FI generators* $H_0^{\text{FI}}(\mathcal{V})$ of \mathcal{V} are a sequence of symmetric group representations that we think of as encoding the “unstable” elements of \mathcal{V} . In degree k , the \mathfrak{S}_k -representation $H_0^{\text{FI}}(\mathcal{V})_k$ is defined to be the cokernel

$$H_0^{\text{FI}}(\mathcal{V})_k := \text{cokernel} \left(\bigoplus_{a \in [k]} \mathcal{V}_{[k] \setminus \{a\}} \rightarrow \mathcal{V}_k \right)$$

where the maps are induced by the natural inclusions $[k] \setminus \{a\} \hookrightarrow [k]$. We think of elements in the kernel of the natural surjection $\mathcal{V} \rightarrow H_0^{\text{FI}}(\mathcal{V})$ as being “stable;” these are all elements of \mathcal{V}_k in the image of some lower degree component \mathcal{V}_j , $j < k$.

We say that an FI-module \mathcal{V} is *generated in degree $\leq d$* (or has *generation degree $\leq d$*) if $H_0^{\text{FI}}(\mathcal{V})_k \cong 0$ for $k > d$. We say that \mathcal{V} is *finitely generated* if $\bigoplus_{k \geq 0} H_0^{\text{FI}}(\mathcal{V})_k$ is finitely generated as an R -module. Finite generation is equivalent to the condition that there is a finite subset of $\bigoplus_{k \geq 0} \mathcal{V}_k$ whose images under the FI morphisms generate $\bigoplus_{k \geq 0} \mathcal{V}_k$ as an R -module. Church, Ellenberg, Farb, and Nagpal [CEF15, CEFN14] proved many consequences of finite generation, notably, if \mathcal{V} is a finitely generated FI-module and R is a field of characteristic zero, then the multiplicities of the irreducible constituents of \mathcal{V}_k eventually stabilize [CEF15, Theorem 1.13]. They also proved that when R is a field, the dimensions $\dim_R(\mathcal{V}_k)$ are (for k sufficiently large) equal to a polynomial in k [CEFN14, Theorem B].

Stability in the homology of configuration spaces

Given a noncompact manifold M , for each i , the i th homology groups $\{H_i(F_k(M))\}_{k=0}^\infty$ of the configuration spaces have the structure of an FI-module, denoted $H_i(F(M))$, which we now describe. We take homology with coefficients in a fixed commutative, unital ring R unless otherwise stated. Given a finite set S , let $F_S(M)$ denote the space of embeddings of S into M . If $|S| = k$, a choice of bijection $S \cong [k]$ gives a homeomorphism between $F_S(M)$ and $F_k(M)$. Every injective map of sets $f : S \hookrightarrow T$ defines a map $\bar{f} : F_S(M) \rightarrow F_T(M)$, as in Figure 4. We use the injection $S \hookrightarrow T$ to relabel the configuration,

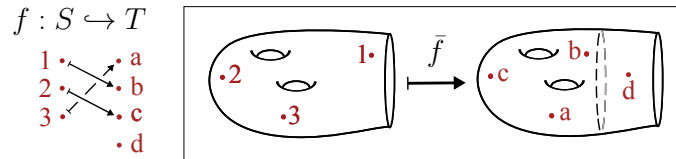


Figure 4: The FI-module structure on $H_i(F(M); R)$.

and elements of $T \setminus f(S)$ to label particles in the image of $\mathbb{R}^n \hookrightarrow M$.

Although the map f depends on many choices, up to homotopy it only depends on the isotopy class of the embedding e and the injection $S \hookrightarrow T$. In the language of FI-modules, Theorem 1.1 is the statement that $H_0^{\text{FI}}(H_i(F(M)))_S$ vanishes when $|S| > 2i$. If M has finite type then the FI-module $H_i(F(M))$ is finitely generated in degree $\leq 2i$. For $k \geq 2i$, every homology class in $H_i(F_k(M))$ is an R -linear combination of homology classes of the form of Figure 5: there are at most $2i$ particles moving around M in an i -parameter family, and the remaining particles stand still near the boundary. Figure

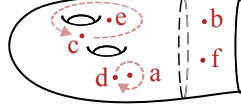


Figure 5: A stable homology class in $H_2(F(M))_{\{a,b,c,d,e,f\}}$.

5 is the image of an FI generator under the FI morphism

$$\{a, c, d, e\} \hookrightarrow \{a, b, c, d, e, f\}.$$

Church, Ellenberg, and Farb showed that the homology groups of ordered configuration spaces are *free* FI-modules in the sense of Definition 2.15 [CEF15, Definition 4.1.1 and Section 6.4]. Free FI-modules contain their FI generators as subrepresentations, and all FI morphisms act by injective maps. Moreover, a free FI-module is completely determined by its generators (see Theorem 2.16, quoting [CEF15, Theorem 4.1.5]). Since the groups $H_0^{\text{FI}}(H_i(F(M)))_k$ determine all homology groups of $F_k(M)$, the objective of this paper is to achieve a better understanding of these unstable homology groups. In particular we will establish patterns relating these groups in different homological degrees.

The FI-module maps $H_i(F_k(M)) \rightarrow H_i(F_{k+1}(M))$ have an induced action on the quotient space of FI generators, but by construction all morphisms act by the zero map. Typically there are no other natural maps from $H_0^{\text{FI}}(H_i(F(M)))_k$ to $H_0^{\text{FI}}(H_i(F(M)))_{k+1}$. However, there are natural maps:

$$H_0^{\text{FI}}\left(H_i(F(M))\right)_k \longrightarrow H_0^{\text{FI}}\left(H_{i+1}(F(M))\right)_{k+2}.$$

In this paper we study this new operation: instead of stabilizing by introducing a single stationary particle near the boundary, we stabilize by introducing two orbiting particles.

Secondary representation stability

Given $i \geq 0$ and a finite set S , let $\mathcal{W}_i^M(S)$ be the sequence of unstable homology groups

$$\mathcal{W}_i^M(S) = H_0^{\text{FI}}\left(H_{\lfloor \frac{|S|}{2} + i} (F(M); R)\right)_S.$$

By convention, fractional homology groups are zero. Any injection $S \hookrightarrow T$ with $|T| - |S| = 2$ induces a map $\mathcal{W}_i^M(S) \rightarrow \mathcal{W}_i^M(T)$ as shown in Figure 6.

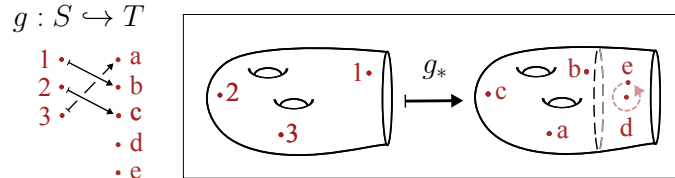


Figure 6: Stabilization by orbiting particles.

If $|T| - |S| = 2d$ for $d > 1$, the data of the injection is not enough to define a map $\mathcal{W}_i^M(S) \rightarrow \mathcal{W}_i^M(T)$. In addition to the injection $f: S \hookrightarrow T$, we choose a *perfect matching* on the complement $T \setminus f(S)$, that

is, a partition of $T \setminus f(S)$ into d sets of size 2. This matching determines how the particles will pair off. To specify the sign of the resultant homology class, we then choose an *orientation* on the perfect matching (see Definition 2.9). We define a stabilization map on $F_S(M)$ by introducing these d pairs of orbiting particles near the boundary, as in Figure 7.

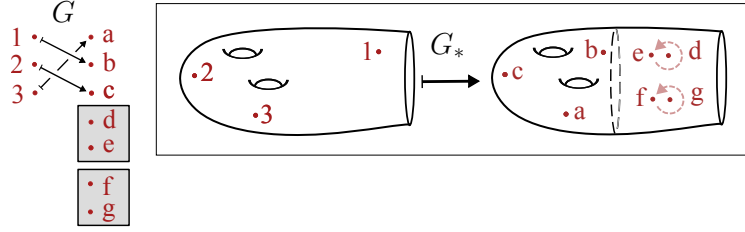


Figure 7: The $\bigwedge(\text{Sym}^2 R)$ -module structure on $H_0^{\text{FI}}(H_*(F(M); R))$.

These operations and the symmetric group actions give the sequences \mathcal{W}_i^M the structure of modules over the *twisted skew-commutative algebra* $\bigwedge(\text{Sym}^2 R)$, or, equivalently, a module over the enriched category FIM^+ of Definition 2.9. See work of Nagpal, Sam, and Snowden [SS12, SS15, NSS16a, NSS16b] and Section 2.1 for more information on twisted (skew-)commutative algebras. In this language, Theorem 1.2 can be formulated as follows.

Theorem 1.4. *If R is a field of characteristic zero and M is a connected noncompact manifold of finite type and dimension at least two, then for each $i \geq 0$ the sequence of unstable homology groups \mathcal{W}_i^M are finitely generated as a $\bigwedge(\text{Sym}^2 R)$ -module.*

We call this finite generation result *secondary representation stability*. This implies that there is some number N_i such that the FI generators $H_0^{\text{FI}}(H_{i+\frac{k}{2}}(F(M)))_k$ are spanned by classes of the form of Figure 8, where all but at most N_i many particles move in orbiting pairs near the boundary. For

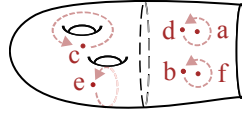


Figure 8: A secondary stable class in $H_0^{\text{FI}}(H_4(F(M)))_{\{a,b,c,d,e,f\}}$.

orientable surfaces, representation stability [CEF15, Theorem 6.4.3] is shown graphically in Figure 9, and secondary representation stability in Figure 10.

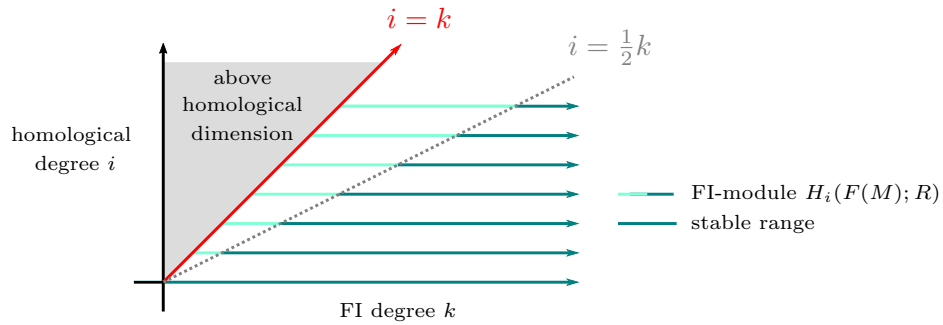


Figure 9: The FI-modules $H_i(F(M); R)$ for an orientable surface M .

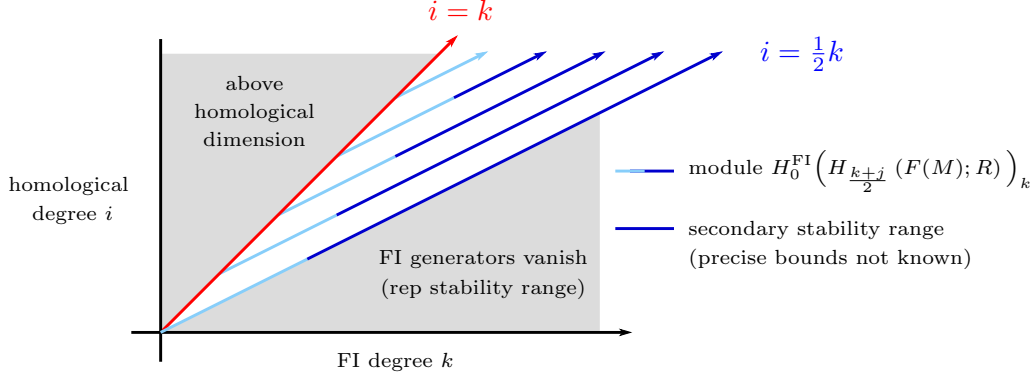


Figure 10: The FI generators $H_0^{\text{FI}}(H_i(F(M)))_k$ for an orientable surface M .

Viewing these homology groups as a $\bigwedge(\text{Sym}^2 R)$ -module and drawing on the theory of twisted skew-commutative algebras, we can prove a version of the main theorem that establishes isomorphisms instead of just surjections.

Corollary 1.5. *Let R be a field of characteristic zero. For k sufficiently large compared to i , $\mathcal{W}_i^M(k)$ is isomorphic to the coequalizer of two natural maps:*

$$\text{Ind}_{\mathfrak{S}_{k-4} \times \mathfrak{S}_2 \times \mathfrak{S}_2}^{\mathfrak{S}_k} \mathcal{W}_i^M(k-4) \rightrightarrows \text{Ind}_{\mathfrak{S}_{k-2} \times \mathfrak{S}_2}^{\mathfrak{S}_k} \mathcal{W}_i^M(k-2).$$

Concretely, this says that $H_0^{\text{FI}}\left(H_{\frac{k+i}{2}}(F(M); R)\right)_k$ is the coequalizer of the maps:

$$\text{Ind}_{\mathfrak{S}_{k-4} \times \mathfrak{S}_2 \times \mathfrak{S}_2}^{\mathfrak{S}_k} H_0^{\text{FI}}\left(H_{\frac{k+i}{2}-2}(F(M); R)\right)_{k-4} \rightrightarrows \text{Ind}_{\mathfrak{S}_{k-2} \times \mathfrak{S}_2}^{\mathfrak{S}_k} H_0^{\text{FI}}\left(H_{\frac{k+i}{2}-1}(F(M); R)\right)_{k-2}.$$

In particular, the representations $\mathcal{W}_i^M(k-4)$ and $\mathcal{W}_i^M(k-2)$ together with the maps $\mathcal{W}_i^M(k-2) \rightarrow \mathcal{W}_i^M(k-4)$ completely determine the representations $\mathcal{W}_i^M(k)$ in the stable range. This corollary can be viewed as a secondary version of *central stability* in the sense of Putman [Put15].

If M is at least three dimensional, then the homology class of two particles circling each other vanishes. This implies that the maps

$$\text{Ind}_{\mathfrak{S}_{k-2} \times \mathfrak{S}_2}^{\mathfrak{S}_k} \mathcal{W}_i^M(k-2) \rightarrow \mathcal{W}_i^M(k)$$

are zero. On the other hand, in the secondary stability range, these maps are surjections, so $\mathcal{W}_i^M(k)$ vanishes for k sufficiently large. For higher dimensional manifolds M , then, secondary stability is the statement that $H_i(F_k(M))$ is representation stable in an improved range, and in Theorem 3.27 we prove explicit stability bounds for these homology groups with integral coefficients.

For surfaces, however, the groups \mathcal{W}_i^M are generally nonzero. For example, $\mathcal{W}_i^{\mathbb{R}^2}(2k+i)$ is a free abelian group whose rank grows super-exponentially in k ; see Proposition 3.34. In Section 3.6, we formulate some conjectures for tertiary and higher order stability.

Remark 1.6. Although we only consider configuration spaces in this paper, we expect secondary representation stability may be present in other situations. For example, it would be worth investigating if similar a phenomenon is present in the homology of congruence subgroups of classical linear groups.

The proof of secondary stability and some possible generalizations

The proof of Theorem 1.4 involves the analysis of a semi-simplicial space, the *arc resolution* of $F_k(M)$, described in Section 3.2. In Section 3.3, we compute certain differentials in spectral sequences associated to the arc resolutions, which we use to prove the desired finiteness properties of the sequences $\mathcal{W}_k^i(M)$ in Section 3.4. The algebraic underpinnings of our proof of secondary representation stability is

developed in Section 2, and draws on the theory of FI-modules introduced by Church, Ellenberg, and Farb [CEF15], the central stability complex introduced by Putman in [Put15], and the theory of twisted skew-commutative algebras. In particular, our proof relies on the Noetherian property for $\bigwedge(\mathrm{Sym}^2 R)$ -modules established by Nagpal, Sam and Snowden [NSS16b, Theorem 1.1].

This Noetherian property for $\bigwedge(\mathrm{Sym}^2 R)$ -modules is currently only known when R is a field of characteristic zero. If it were possible to prove this result over more general commutative unital rings R , then (with a modification of our Proposition 3.24) our proof would establish our main results, Theorem 1.2, Theorem 1.4, and Corollary 1.5, over these rings. Additionally, if one could bound the degrees of higher syzygies of $\bigwedge(\mathrm{Sym}^2 R)$ -modules as Church and Ellenberg do for FI-modules [CE15, Theorem A], it would be possible to explicitly bound the function r from Theorem 1.2.

Some conjectural generalizations and strengthenings of Theorem 1.4 are discussed in Section 3.6.

1.3 Other results

In the process of establishing secondary representation stability for configuration spaces, we prove some other results which may be of independent interest. In particular, we prove new representation stability results for the homology and cohomology of configuration spaces, and we give a new Lie-theoretic description of the top homology group of the complex of injective words.

The homology of the complex of injective words

The *complex of injective words* $\mathrm{Inj}_\bullet(k)$ on the set $[k]$ is a semi-simplicial set which was used by Kerz [Ker05] to give a new proof of homological stability for the symmetric groups (see Definition 2.17). It has found application in algebraic topology, representation theory, and algebraic combinatorics. The complex of injective words has only one nonvanishing reduced homology group, a subgroup of the free abelian group on the set of k -letter words on the set $[k]$. In Section 2.3, we describe an explicit basis for this group that resembles the Poincaré–Birkhoff–Witt basis for the free Lie superalgebra on $[k]$.

Theorem 2.43. *The reduced integral homology group $\tilde{H}_{k-1}(|\mathrm{Inj}_\bullet(k)|)$ is the submodule of the free associative algebra on the set $[k]$ generated by products of iterated graded commutators where every element of $[k]$ appears exactly once. An explicit basis for this free abelian group is given in Lemma 2.41.*

For example, the signed sum of injective words given by expanding the product of graded commutators $[[1, 2], 4] [3, 5]$ is an element of $\tilde{H}_4(|\mathrm{Inj}_\bullet(5)|)$.

Primary representation stability for configuration spaces

The work of Church, Ellenberg, Farb, and Nagpal [Chu12, CEF15, CEFN14] on representation stability for configuration spaces uses Totaro’s spectral sequence [Tot96], which assumes that the manifold is orientable. We remove this assumption by giving an entirely different proof of representation stability for configuration spaces. Following methods of Putman [Put15] on congruence subgroups, we adapt Quillen’s approach to homological stability to prove representation stability. Rationally, this result was previously proven in the unpublished work of Søren Galatius (see Palmer [Pal13, Remark 1.11]) using different techniques (also see Casto [Cas16, Corollary 3.3]). Our results are given in Theorem 3.11 for general noncompact manifolds and Theorem 3.27 for manifolds of dimension at least three.

In Appendix A, we build on the work of Church and Ellenberg [CE15] on FI-homology to give the first quantitative bounds for integral representation stability for the cohomology of configuration spaces of particles in *closed* manifolds in Theorem A.12.

Theorems 3.11, 3.27, and A.12. *Let M be a connected manifold of dimension $n \geq 2$.*

- (a) *Suppose M is noncompact. Then $H_0^{\mathrm{FI}}(H_i(F(M)))_k \cong 0$ for $k > 2i$.*
- (b) *Suppose M is noncompact and has dimension at least 3. Then $H_0^{\mathrm{FI}}(H_i(F(M)))_k \cong 0$ for $k > i$.*
- (c) *Suppose only that M has finite type. Then the FI-module $H_0^{\mathrm{FI}}(H^i(F(M)))_k \cong 0$ if $k > 21(i + 1)(1 + \sqrt{2})^{i-2}$ and $H_1^{\mathrm{FI}}(H^i(F(M)))_k \cong 0$ if $k > 28(i + 1)(1 + \sqrt{2})^{i-2}$.*

Here H_1^{FI} denotes the first left-derived functor of H_0^{FI} . By Theorem A of [CE15], Theorem A.12 can be used to prove vanishing in a range for all higher derived functors of H_0^{FI} of the cohomology groups of configuration spaces of compact manifolds.

1.4 Acknowledgments

Our project was inspired by results on *secondary homological stability* by Søren Galatius, Alexander Kupers, and Oscar Randal-Williams. Using the theory of E_n -cells, these authors have established secondary homological stability in many examples including classifying spaces of general linear groups and mapping class groups [GKRW]. Our project benefited greatly from our interaction with them.

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2 Algebraic foundations

The goal of this section is to lay the algebraic groundwork necessary to state and prove the main theorem. We begin, in Section 2.1, with a review of FI-modules and their generalizations, modules over a twisted (skew-)commutative algebra. This provides a very general context for formulating representation stability for sequences of symmetric group representations. We then discuss the relationship between Putman’s central stability chain complex [Put15] and Farmer’s complex of injective words [Far79] in Section 2.2. In Section 2.3, we give a new description of the homology of the complex of injective words. In Section 2.4, we conclude with an analysis of a generalization of the central stability chain complex for FIM^+ -modules. These chain complexes will appear in Section 3 on the pages of the arc resolution spectral sequence, a spectral sequence we use to prove secondary stability for configurations spaces.

2.1 Review of twisted (skew-)commutative algebras

Throughout this paper, we fix a commutative unital ring R . All homology groups will be assumed to have coefficients in R , all tensor products will be taken over R , and so forth, unless otherwise specified.

Church and Farb [CF13] introduced the concept of representation stability for sequences of representations of groups whose representation theory is well-behaved and well-understood. Church, Ellenberg, and Farb [CEF15] reinterpreted representation stability for sequences of symmetric group representations in terms of modules over the category FI. They used the following notation, which we now adopt.

Definition 2.1. Let FI be the category whose objects are finite (possibly empty) sets and whose morphisms are injective maps. Let FB be the category of finite sets and bijective maps.

Definition 2.2. Let \mathcal{C} be a category. A \mathcal{C} -module (over a ring R) is a covariant functor from \mathcal{C} to $R\text{-Mod}$, the category of (left) R -modules. A *co- \mathcal{C} -module* (over R) is a contravariant functor from \mathcal{C} to $R\text{-Mod}$.

Recall that we denote the value of an FB or FI-module \mathcal{V} on a set S by \mathcal{V}_S (or possibly $\mathcal{V}(S)$ in instances where \mathcal{V} has other subscripts). When S is the set $[k] = \{1, 2, \dots, k\}$, we write \mathcal{V}_k or $\mathcal{V}(k)$.

Remark 2.3. The category of FB-modules over R is equivalent to the category of *symmetric sequences* of R -modules, that is, the category of nonnegatively graded R -modules V_* such that V_k has an action of \mathfrak{S}_k . We sometimes view FB-modules as symmetric sequences by restricting to the skeletal subcategory of FB of finite sets of the form $[k]$ for $k \in \mathbb{N}_0$. We will similarly sometimes restrict FI to this skeleton.

We can generalize Definition 2.2 by replacing the category of R -modules by other categories.

Definition 2.4. A \mathcal{C} -space is a covariant functor from \mathcal{C} to the category of topological spaces. A *homotopy \mathcal{C} -space* is a covariant functor from \mathcal{C} to the homotopy category of topological spaces. (Homotopy) co- \mathcal{C} -spaces are the corresponding contravariant functors.

The category of FI-modules studied by Church, Ellenberg, and Farb [CEF15] was later understood to be an example of a category of modules over a twisted commutative algebra (tca). We will use the theory of (skew-)tca's to define secondary representation stability, and we summarize the key aspects of this theory here. See the work of Church, Ellenberg, Farb, and Nagpal [CEF15, CEFN14, CE15] for more results on FI-modules, and see work of Nagpal, Sam, and Snowden [Sno13, SS15, SS12, NSS16a, NSS16b] for more results on twisted (skew-)commutative algebras.

Definition 2.5. A *twisted commutative algebra* is an FB-module \mathcal{A} equipped with a multiplication

$$\mathcal{A}_S \otimes \mathcal{A}_T \longrightarrow \mathcal{A}_{S \sqcup T}$$

which is functorial, associative, has unit $1 \in \mathcal{A}_\emptyset$, and satisfies the following condition:

- (*) Every $x \in \mathcal{A}_S$ and $y \in \mathcal{A}_T$ satisfy $yx = \tau(xy)$, where $\tau : S \sqcup T \rightarrow T \sqcup S$ is the canonical isomorphism.

The definition of a *twisted skew-commutative algebra* is the same as the above after axiom (*) is replaced by a skew-commutative variant (**):

- (**) Every $x \in \mathcal{A}_S$ and $y \in \mathcal{A}_T$ satisfy $yx = (-1)^{|S||T|} \tau(xy)$, where $\tau : S \sqcup T \rightarrow T \sqcup S$ is the canonical isomorphism.

A *module over a twisted (skew-)commutative algebra* \mathcal{A} is an FB-module \mathcal{V} with an action

$$\mathcal{A}_S \otimes \mathcal{V}_T \rightarrow \mathcal{V}_{S \sqcup T}$$

satisfying the appropriate functoriality, associativity, and unity axioms.

The details of this definition are given in [SS12, 8.1.5]. See [SS12, Section 8] for several equivalent definitions of a tca.

Remark 2.6. The category of symmetric sequences has a symmetric monoidal product sometimes called the *convolution product* or the *induction product*; see for example Fresse [Fre00, Definitions 1.1.1 and 1.1.4]. We could equivalently define tca's as commutative monoids with respect to this product.

We will primarily be interested in the following examples of twisted (skew-)commutative algebras.

Definition 2.7. Let TCA denote category of twisted commutative algebras over R , and let STCA denote the category twisted skew-commutative algebras over R . There are forgetful functors from TCA and STCA to FB-Mod. Let

$$\text{Sym} : \text{FB-Mod} \longrightarrow \text{TCA} \quad \text{and} \quad \bigwedge : \text{FB-Mod} \longrightarrow \text{STCA}$$

be the left adjoints of these forgetful functors. Let $\text{Sym}^m R$ denote the FB-module

$$(\text{Sym}^m R)_k = \begin{cases} 0, & k \neq m \\ \text{trivial } \mathfrak{S}_m\text{-representation } R, & k = m. \end{cases}$$

The tca $\text{Sym}(\text{Sym}^1 R)$ is the FB-module with a rank-1 trivial S_k -representation R in every degree, and all multiplication maps given by the canonical isomorphisms $R \otimes R \cong R$. The data of a module over $\text{Sym}(\text{Sym}^1 R)$ is equivalent to an FI-module \mathcal{V} over R . See Sam–Snowden [SS12, Section 10.2].

The tca $\text{Sym}(\text{Sym}^2 R)$ is generated by $\text{Sym}(\text{Sym}^2 R)_{\{a,b\}} \cong R\langle x_{a,b} \mid x_{a,b} = x_{b,a} \rangle$. The multiplication map is given by the usual multiplication of (commutative) monomials in the variables $x_{a,b}$; the indices of these monomials are all distinct by construction. Modules over $\text{Sym}(\text{Sym}^2 R)$ are equivalent to modules over the combinatorial category FIM we now define (see also [SS15, Section 4.3]).

Definition 2.8. A *matching* of a set B is a set of disjoint 2-element subsets B , and a matching is a *perfect matching* if the union of these subsets is B . Let FIM be the category whose objects are finite sets and whose morphisms are injective maps $f : S \hookrightarrow T$ together with the data of a perfect matching of the complement $T \setminus f(S)$ of the image. Composition of morphisms is defined by composing injective maps and taking the union of one matching with the image of the other.

The skew-tca $\bigwedge(\text{Sym}^2 R)$ is generated by $\bigwedge(\text{Sym}^2 R)_{\{a,b\}} \cong R\langle x_{a,b} \mid x_{a,b} = x_{b,a} \rangle$, with the usual multiplication of anticommutative monomials in $x_{a,b}$ with distinct indices. The category of modules over $\bigwedge(\text{Sym}^2 R)$ cannot be encoded as a functor category to $R\text{-Mod}$, however, $\bigwedge(\text{Sym}^2 R)$ -modules are equivalent to modules over an enriched category we denote FIM^+ .

Definition 2.9. Let FIM^+ be following category enriched over $R\text{-Mod}$. The objects are finite sets. The module of morphisms between sets of different parity is the R -module $\{0\}$. Between sets $[a - 2b]$ and $[a]$, the space of morphism is the following quotient:

$$\frac{R \left\langle (f : [a - 2b] \rightarrow [a], A_1, A_2, \dots, A_b) \mid \begin{array}{l} f \text{ is injective, } |A_i| = 2, \quad [a] = \text{im}(f) \sqcup A_1 \sqcup \dots \sqcup A_b \\ \text{so } \{A_i\} \text{ is an ordered perfect matching on } [a] \setminus \text{im}(f) \end{array} \right\rangle}{\left\langle (f, A_1, A_2, \dots, A_b) = (-1)^\sigma (f, A_{\sigma(1)}, A_{\sigma(2)}, \dots, A_{\sigma(b)}) \text{ for all } \sigma \in \mathfrak{S}_b \right\rangle}$$

In other words, when $k \equiv m \pmod{2}$, the morphisms from $[k]$ to $[m]$ are the free R -module on the set of all injective maps $[k] \hookrightarrow [m]$ along with a perfect matching on the complement of the image, and a choice of orientation on the perfect matching. We denote a free generator of the morphisms by

$$F = (f, A_1 \wedge A_2 \wedge \dots \wedge A_b).$$

The composition of the maps

$$F = (f, A_1 \wedge A_2 \wedge \dots \wedge A_b) \quad \text{and} \quad G = (g, C_1 \wedge C_2 \wedge \dots \wedge C_d)$$

is given by the map

$$G \circ F = (g \circ f, C_1 \wedge C_2 \wedge \dots \wedge C_d \wedge g(A_1) \wedge g(A_2) \wedge \dots \wedge g(A_b)).$$

Definition 2.10. Let \mathcal{C} be a category enriched over $R\text{-Mod}$. We define a \mathcal{C} -module to be an enriched functor from \mathcal{C} to $R\text{-Mod}$.

Recall from the introduction that we define the FI generators of an FI-module \mathcal{V} to be the sequence of \mathfrak{S}_k -representations

$$H_0^{\text{FI}}(\mathcal{V})_k = \mathcal{V}_k / \text{im} \left(\bigoplus_{a \in [k]} \mathcal{V}_{[k] \setminus \{a\}} \right).$$

We now extend this definition to modules over a general (skew-)tca.

Definition 2.11. Let \mathcal{V} be a module over a (skew-)tca \mathcal{A} . Let $H_0^{\mathcal{A}}(\mathcal{V})_S$ be the quotient

$$H_0^{\mathcal{A}}(\mathcal{V})_S := \text{cokernel} \left(\bigoplus_{S=P \sqcup Q, P \neq \emptyset} \mathcal{A}_P \otimes \mathcal{V}_Q \longrightarrow \mathcal{V}_S \right).$$

The R -modules $H_0^{\mathcal{A}}(\mathcal{V})_S$ assemble to form an \mathcal{A} -module with an action of \mathcal{A}_P by trivial maps for $|P| > 0$. We say that \mathcal{V} is *finitely generated* if $\bigoplus_{k=0}^{\infty} H_0^{\mathcal{A}}(\mathcal{V})_k$ is finitely generated as an R -module.

We often replace the superscript \mathcal{A} in the notation $H_0^{\mathcal{A}}(\mathcal{V})$ with the corresponding category. Following Church–Ellenberg [CE15], we use the following terminology.

Definition 2.12. Let \mathcal{V} be an \mathcal{A} -module with \mathcal{A} a (skew-)tca. We say that $\deg \mathcal{V} \leq d$ if $\mathcal{V}_k = 0$ for all $k > d$. We say \mathcal{V} is generated in degrees $\leq d$ if $\deg H_0^{\mathcal{A}}(\mathcal{V}) \leq d$.

Definition 2.13. Let \mathcal{A} be a (skew-)tca. We define $M^{\mathcal{A}}$ to be the left adjoint to the forgetful functor $\mathcal{A}\text{-Mod} \rightarrow \text{FB-Mod}$. We call modules in the image of $M^{\mathcal{A}}$ *free* \mathcal{A} -modules. Given an $R[\mathfrak{S}_d]$ -module W , we define $M^{\mathcal{A}}(W)$ by viewing W as the FB-module with module W in degree d and 0 in all other degrees. We let $M^{\mathcal{A}}(d) := M^{\mathcal{A}}(R[\mathfrak{S}_d])$. We will often replace the superscript \mathcal{A} with its corresponding category, and (following [CEF15, Definition 2.2.2]) simply write M in the case when the category is FI.

To prove the main results in this paper, we will construct resolutions of finitely generated $\bigwedge (\text{Sym}^2 R)$ -modules by free modules which are generated in finite degree. To do this we use the following Theorem of Napol, Sam, and Snowden [NSS16b, Theorem 1.1].

Theorem 2.14 (Nagpal–Sam–Snowden [NSS16b, Theorem 1.1]). *Let R be a field of characteristic zero. Any submodule of a finitely generated module over $\bigwedge (\text{Sym}^2 R)$ is finitely generated.*

This Noetherian property also holds for rational FI-modules; this result was proved by Snowden [Sno13, Theorem 2.3] and later, independently, by Church, Ellenberg, and Farb [CEF15, Theorem 1.3]. Church, Ellenberg, Farb, and Nagpal later proved that this Noetherian property in fact holds over any Noetherian coefficient ring [CEFN14, Theorem A]. We will not need these results on FI-modules, however, since the FI-modules given by the homology of the configuration spaces of a noncompact manifold are free in the sense of Definition 2.13.

Church, Ellenberg, and Farb showed that, given an \mathfrak{S}_d -representation W , the free FI-module $M(W)$ satisfies

$$M(W)_k \cong \bigoplus_{\substack{A \subseteq [k] \\ |A|=d}} W \cong \text{Ind}_{\mathfrak{S}_d \times \mathfrak{S}_{k-d}}^{\mathfrak{S}_k} W \boxtimes R$$

where R denotes the trivial \mathfrak{S}_{k-d} -representation. For a general FB-module W , these FI-modules satisfy

$$H_0^{\text{FI}}(M(W)) \cong W.$$

These authors prove that the free FI-modules $M(W)$ can be promoted to modules over the larger category $\text{FI}_\#$, which we define as follows.

Definition 2.15. Define a *based injection* $f : S_0 \rightarrow T_0$ between two based sets S_0, T_0 to be a based map such that $|f^{-1}(\{a\})| \leq 1$ for all elements $a \in T_0$ except possibly the basepoint. Let $\text{FI}_\#$ be the category whose objects are finite based sets and whose morphisms are based injections.

The category defined in Definition 2.15 is isomorphic to the category called $\text{FI}_\#$ by Church, Ellenberg, and Farb [CEF15, Definition 4.1.1]. The operation of adding a basepoint gives an embedding of categories $\text{FI} \subseteq \text{FI}_\#$. Hence an $\text{FI}_\#$ -module is an FI-module with additional structure and constraints, notably, the FI morphisms have one-sided inverses and so must act by injective maps. These backwards maps give $\text{FI}_\#$ -modules the structure of co-FI-modules, and we may view $\text{FI}_\#$ -modules as co-FI-modules with a compatible FI-module structure. The following result of Church, Ellenberg, and Farb gives a classification of $\text{FI}_\#$ -modules: they are precisely the free FI-modules. They show moreover that the functors $M : \text{FB-Mod} \rightarrow \text{FI}_\#\text{-Mod}$ and $H_0^{\text{FI}} : \text{FI}_\#\text{-Mod} \rightarrow \text{FB-Mod}$ are inverses, and define an equivalence of categories.

Theorem 2.16 ([CEF15, Theorem 4.1.5]). *An FI-module \mathcal{V} is the restriction of an $\text{FI}_\#$ -module if and only if it is free, in which case it is the restriction of a unique $\text{FI}_\#$ -module. In particular, for an $\text{FI}_\#$ -module \mathcal{V} , there is an isomorphism*

$$\mathcal{V} \cong \bigoplus_{k=0}^{\infty} M(H_0^{\text{FI}}(\mathcal{V})_k).$$

Notably, Theorem 2.16 implies that an $\text{FI}_\#$ -module \mathcal{V} is completely determined by its FI generators.

2.2 Twisted injective word complexes

Putman [Put15] defined a chain complex associated to a sequence of \mathfrak{S}_k -representations called the *central stability chain complex*. This chain complex arises as the E^1 -page of a certain spectral sequence, and its homology is the E^2 -page. Natural analogues of the chain complex exist when the symmetric groups are replaced by other families of groups such as general linear groups. See for example the work of Putman and Sam [PS14, Section 5.3]. In the context of FI-modules, we show that this chain complex is closely related to the complex of injective words and accordingly we will denote the complex using the notation Inj . To define this complex, we first recall the definition of the complex of injective words. In this subsection, we work with integral coefficients.

Definition 2.17. For a set S and an integer $i \geq -1$, let $\text{Inj}_i(S) = \text{Hom}_{\text{FI}}(\{0, \dots, i\}, S)$.

For a fixed set S , $\text{Inj}_\bullet(S)$ has the structure of an augmented semi-simplicial set. The face map d_j acts by precomposition with the order-preserving injective map $\{0, \dots, i-1\} \rightarrow \{0, \dots, i\}$ that misses the element j . We define $\text{Inj}_{-1}(S)$ to be a single point.

Farmer [Far79] proved the following result on the connectivity of $||\text{Inj}_\bullet(S)||$.

Theorem 2.18 (Farmer [Far79]). *The geometric realization $||\text{Inj}_\bullet(S)||$ is $|S| - 2$ connected.*

This fact can be used to prove homological stability for the symmetric groups; see Kerz [Ker05]. Since $||\text{Inj}_\bullet(S)||$ has dimension $|S| - 1$, the reduced homology of $||\text{Inj}_\bullet(S)||$ is concentrated in dimension $|S| - 1$. We now recall Putman's central stability chain complex, which we view as a twisted versions of the complex of injective words. We generalize Farmer's connectivity result.

Definition 2.19. For a set S , an FI-module \mathcal{V} , and integer $i \geq -1$, let

$$\text{Inj}_i(\mathcal{V})_S = \bigoplus_{f: \{0, \dots, i\} \hookrightarrow S} \mathcal{V}_{S \setminus \text{im}(f)}.$$

These groups assemble into an augmented semi-simplicial FI-module $\text{Inj}_\bullet(\mathcal{V})$. Let $\text{Inj}_*(\mathcal{V})$ denote the associated FI-chain complex. When \mathcal{V} is the FI-module $M(0)$, for a set S the complex $\text{Inj}_*(\mathcal{V})_S$ is precisely the chain complex associated to the augmented semi-simplicial set $\text{Inj}_\bullet(S)$.

Remark 2.20. Given an FI-module \mathcal{V} , the chain complex $\text{Inj}_*(\mathcal{V})$ is denoted by $IA_n(\mathcal{V})$ in Section 4 of [Put15], by $\tilde{S}_\bullet(\mathcal{V})$ in Definition 2.19 of [CEFN14] and by $\Sigma_*(\mathcal{V})$ in Section 3 of [PS14]. We apologize for adding yet another name for this chain complex.

The next goal is to compute the homology of this chain complex on $\text{FI}^\#$ -modules.

Remark 2.21. Suppose that \mathcal{V} is an FI-module such that $\mathcal{V}_k = 0$ for all $k < d$. Observe that by Definition 2.19, $\text{Inj}_i(\mathcal{V})_S = 0$ whenever $|S| - i - 1 < d$.

Remark 2.22. It follows from the definition of $\text{Inj}_i(\mathcal{V})_k$ that there is an isomorphism of \mathfrak{S}_k -representations

$$\text{Inj}_i(\mathcal{V})_k \cong \text{Ind}_{\mathfrak{S}_{k-i-1}}^{\mathfrak{S}_k} \mathcal{V}_{k-i-1}.$$

In particular, for the FI-module $M(d)$ there is an isomorphism of \mathfrak{S}_k -representations

$$\text{Inj}_i(M(d))_k \cong M(d + i + 1)_k.$$

Given an \mathfrak{S}_d -representation W , there is an isomorphism of \mathfrak{S}_k -representations

$$\text{Inj}_i(M(W))_k \cong M\left(\text{Ind}_{\mathfrak{S}_d}^{\mathfrak{S}_{d+i+1}} W\right)_k.$$

Definition 2.23. For a set S , an integer $d \geq 0$ and an integer $i \geq -1$, let:

$$\text{Inj}_i(d)(S) = \bigsqcup_{f \in \text{Hom}_{\text{FI}}(\{0, \dots, i\}, S)} \text{Hom}_{\text{FI}}([d], S - f(\{0, \dots, i\})).$$

$\text{Inj}_\bullet(d)(S)$ has the structure of an augmented semi-simplicial set. For $d = 0$, it is the usual complex of injective words.

Lemma 2.24. *There is an isomorphism:*

$$H_{*+1}(\text{Inj}_{-1}(d)(S), ||\text{Inj}_\bullet(d)(S)||) \cong H_*(\text{Inj}_*(M(d)))_S.$$

Proof. The relative cellular chains on the pair $(\text{Inj}_{-1}(d)(S), ||\text{Inj}_\bullet(d)(S)||)$ is isomorphic to a shift of $\text{Inj}_*(M(d))_S$. \square

Lemma 2.25. *The geometric realization $||\text{Inj}_\bullet(d)(S)||$ is homeomorphic to:*

$$\bigsqcup_{g \in \text{Hom}_{\text{FI}}([d], S)} ||\text{Inj}_\bullet(S - \text{im}(g))||$$

Proof. This homeomorphism is induced from an isomorphism of underlying semi-simplicial sets, as follows. By definition, an element of the p -simplices $\text{Inj}_p(d)(S)$ is a map $f : \{0, \dots, p\} \hookrightarrow S$ and a map $g : \{1, \dots, d\} \hookrightarrow S$ whose images are disjoint. The face maps act by

$$d_j : (g, f) \mapsto (g, f|_{\{0, \dots, \hat{j}, \dots, p\}}).$$

Thus $\text{Inj}_p(d)(S) \cong \bigsqcup_{g: [d] \hookrightarrow S} \text{Hom}_{\text{FI}}(\{0, 1, \dots, p\}, S - \text{img})$, and $\text{Inj}_\bullet(d)(S)$ is isomorphic to the semi-simplicial set

$$\bigsqcup_{g: [d] \hookrightarrow S} \text{Inj}_\bullet(S - \text{img})$$

as claimed. \square

Theorem 2.26. *Let W be an integral representation of \mathfrak{S}_d . There is an isomorphism:*

$$H_i(\text{Inj}_*(M(W)))_S \cong \left(H_i(\text{Inj}_*(M(d))) \otimes_{\mathbb{Z}[\mathfrak{S}_d]} W \right)_S$$

In general, given an $\text{FI}_d^\#$ -module \mathcal{V} ,

$$H_p(\text{Inj}_*(\mathcal{V}))_k = \text{Ind}_{\mathfrak{S}_{p+1} \times \mathfrak{S}_{k-p-1}}^{\mathfrak{S}_k} H_p(\text{Inj}_*(p+1)) \boxtimes (H_0^{\text{FI}}(\mathcal{V}))_{k-p-1}.$$

Proof. Recall that $M(W) \cong M(d) \otimes_{\mathbb{Z}[\mathfrak{S}_d]} W$. Then

$$\text{Inj}_*(M(W))_S \cong \text{Inj}_*(M(d))_S \otimes_{\mathbb{Z}[\mathfrak{S}_d]} W.$$

The homological Künneth spectral sequence (see for example Theorem 10.90 of Rotman [Rot08]), is a first quadrant spectral sequence:

$$E_{p,q}^2 = \text{Tor}_p^{\mathbb{Z}[\mathfrak{S}_d]} \left(H_q \left(\text{Inj}_*(M(d)) \right)_S, W \right).$$

Since the $\mathbb{Z}[\mathfrak{S}_d]$ -modules $\text{Inj}_q(M(d))_S$ are flat, the spectral sequence converges to $H_{p+q}(\text{Inj}_*(M(W)))_S$. Theorem 2.18, Lemma 2.24, Lemma 2.25, and the fact that the map

$$||\text{Inj}_\bullet(d)(S)|| \rightarrow \text{Inj}_{-1}(d)(S)$$

induces a bijection on connected components for $|S| > d$ imply that $E_{p,q}^2 = 0$ except for $q = |S| - 1 - d$. Since the $E_{p,q}^2$ page has only a single nonzero column, the spectral sequence collapses on this page. The limit is nonzero only when $i \geq (|S| - 1 - d)$, and in this case we see that:

$$H_i(\text{Inj}_*(M(W)))_S \cong \text{Tor}_{i-(|S|-1-d)}^{\mathbb{Z}[\mathfrak{S}_d]} \left(H_{|S|-1-d} \left(\text{Inj}_*(M(d)) \right)_S, W \right).$$

On the other hand, $M(W)_k = 0$ for $k < d$, and so by Remark 2.21, $H_i(\text{Inj}_*(M(W)))_S = 0$ whenever $i > |S| - d - 1$.

Thus this spectral sequence has a single nonzero entry. The homology groups $H_i(\text{Inj}_*(M(W)))_S$ are nonzero only in degree $i = |S| - 1 - d$, in which case we have

$$\begin{aligned} H_{|S|-1-d}(\text{Inj}_*(M(W)))_S &\cong \text{Tor}_0^{\mathbb{Z}[\mathfrak{S}_d]} \left(H_{|S|-1-d} \left(\text{Inj}_*(M(d)) \right)_S, W \right) \\ &\cong \left(H_{|S|-1-d} \left(\text{Inj}_*(M(d)) \right) \otimes_{\mathbb{Z}[\mathfrak{S}_d]} W \right)_S. \end{aligned}$$

Theorem 4.1.5 of [CEF15] (here Theorem 2.16) implies that every $\text{FI}_d^\#$ -module is a direct sum of modules of the form $M(W)$. Additionally, for an \mathfrak{S}_d -representation W , $H_0^{\text{FI}}(M(W))_d \cong W$ and $H_0^{\text{FI}}(M(W))_i \cong 0$ for $i \neq d$. These two facts imply the general result. \square

We obtain the following corollary.

Corollary 2.27. *Let \mathcal{V} be an $\mathrm{FI}_\#$ -module with generation degree $\leq d$. Then $H_i(\mathrm{Inj}_*(\mathcal{V}))_S = 0$ for $i \leq |S| - 2 - d$.*

Remark 2.28. For FI -modules \mathcal{V} which are not $\mathrm{FI}_\#$, a version of Theorem 2.27 is also true. The vanishing range for $H_i(\mathrm{Inj}_*(\mathcal{V}))_S$ will depend both on $\deg H_0^{\mathrm{FI}}(\mathcal{V})$ and $\deg H_1^{\mathrm{FI}}(\mathcal{V})$. This proof uses the resolutions by $\mathrm{FI}_\#$ -modules appearing in the proof of Theorem 3.9 of [CE15] to leverage the result for $\mathrm{FI}_\#$ -modules to the case of general FI -modules.

The chain complex $\mathrm{Inj}_*(\mathcal{V})$ is similar to a chain complex used to compute H_i^{FI} . See [CE15, Section 4.1] for a discussion of the relationship between these two chain complexes. In this paper, we only need the following fact relating the two chain complexes. Its proof is immediate from the definitions.

Proposition 2.29. *For any FI -module \mathcal{V} , $H_{-1}(\mathrm{Inj}_*(\mathcal{V}))_S \cong H_0^{\mathrm{FI}}(\mathcal{V})_S$.*

2.3 Homology of the complex of injective words

In the previous subsection, we computed the homology of the injective words chain complex of an $\mathrm{FI}_\#$ -module in terms of the top homology group of the complex of injective words. We now will show this top homology group is a certain space of products of graded Lie polynomials, and compute a basis. In this subsection, we take the ring R to be \mathbb{Z} , however, our computation holds over general commutative unital rings; see Remark 2.44

Throughout this section we let $C_*^{(k)}$ denote the reduced cellular chains on the semi-simplicial space $\mathrm{Inj}_\bullet(k)$. In the language of the previous subsection, $C_*^{(k)} = \mathrm{Inj}_*(M(0))_k$. For $q \geq -1$, the group $C_q^{(k)}$ is the free abelian group on words of $q + 1$ distinct letters in $[k]$. This chain complex has only one nonvanishing homology group, in homological degree $k - 1$.

Definition 2.30. Let $\mathcal{T}_k := H_{k-1}(C_*^{(k)}) \cong \tilde{H}_{k-1}(|\mathrm{Inj}_\bullet(k)|)$.

The symbol \mathcal{T} stands for “top homology group.” Since $C_k^{(k)} = 0$, the homology group \mathcal{T}_k is a submodule of $C_{k-1}^{(k)}$, the kernel of the differential:

$$\mathbf{D} := \sum_{j=0}^{k-1} (-1)^j d_j : C_{k-1}^{(k)} \rightarrow C_{k-2}^{(k)}$$

where d_j is the face map that forgets the j th letter of each word. The top chain group $C_{k-1}^{(k)}$ is naturally isomorphic to the regular representation $\mathbb{Z}[S_k]$, with a \mathbb{Z} -basis given by all injective words on k letters in $[k]$. The main objective of this section is to compute an alternate \mathbb{Z} -basis for $C_{k-1}^{(k)}$ in the style of the Poincaré–Birkhoff–Witt theorem (Theorem 2.38), and identify a sub-basis that spans the kernel of \mathbf{D} (Lemma 2.41 and Theorem 2.43). The result of this calculation is shown explicitly for $k = 2, 3, 4$ in the Example 2.34.

To construct these bases, we draw on the combinatorial theory of Lie superalgebras, also known as graded Lie algebras. We adopt the following notational conventions. If a is a word in the alphabet $[k]$, then in this section we write $|a|$ to mean the word-length of a . If p is a integer linear combination of words, we call p a (*noncommutative*) *polynomial* in $[k]$, and define its degree $|p|$ to be the length of the longest word occurring in p . Polynomials are assumed to be homogeneous unless otherwise stated. For words a and b , we write ab to denote their concatenation; this operation extends linearly to a multiplication on the additive group of polynomials in $[k]$. A word is *injective* if each letter appears at most once. We introduce a graded Lie bracket on polynomials in $[k]$.

Definition 2.31. Define a graded Lie bracket on words in $[k]$ by

$$[a, b] := ab - (-1)^{|a||b|} ba$$

and extend bilinearly to a bracket on the free abelian group on words in $[k]$.

On homogeneous polynomials a, b, c , the Lie bracket satisfies the graded antisymmetry rule

$$[a, b] = -(-1)^{|a||b|}[b, a]$$

and the graded Jacobi identity

$$(-1)^{|a||c|}[a, [b, c]] + (-1)^{|a||b|}[b, [c, a]] + (-1)^{|b||c|}[c, [a, b]] = 0.$$

Definition 2.32. A *Lie polynomial* is any element of the smallest submodule of the free abelian group on words in $[k]$ that contains the elements of $[k]$ and is closed under the Lie bracket.

The space of Lie polynomials is isomorphic to the *free Lie superalgebra on $[k]$* . This space naturally embeds into the free abelian group of words on $[k]$, which, by a graded-commutative version of the Poincaré–Birkhoff–Witt Theorem, we can identify with its universal enveloping algebra. The following result is due to Ross [Ros65, Theorem 2.1]; see also Musson [Mus12, Theorem 6.1.1].

Theorem 2.33 (See, eg, [Ros65, Theorem 2.1]). *Let R be a commutative ring with unit such that 2 is invertible. Let L be a homogeneously free Lie superalgebra over R with homogeneous bases X_0 for its even-graded part and X_1 for its odd-graded part. If \leq is a total order on $X = X_0 \cup X_1$, then the set of monomials of the form*

$$b_1 b_2 \cdots b_m \quad \text{with } b_i \in X, b_i \leq b_{i+1}, \text{ and } b_i \neq b_{i+1} \text{ if } b_i \in X_1$$

and 1 form a free R -basis for the universal enveloping algebra $U(L)$.

We remark that this set of monomials is not a basis when R is \mathbb{Z} . In the case of the free Lie superalgebra on $[k]$, this failure is in some sense due to factors of two that appear with (nested) brackets involving repeated letters, for example, $[1, 1] = 11 + 11$. Fortunately for our purposes, we will show in Theorem 2.38 that those basis elements for which every letter is distinct *do* form an integer basis for C_{k-1}^k . There are also versions of the PBW Theorem in the literature that hold for integer Lie superalgebras, see for example Mikhalev–Zolotykh [MZ95, Theorem 19.1], but this basis for $U(L)$ is not suitable for present purposes.

The following example illustrates the main result of this subsection, the bases for C_{k-1}^k and the top homology group, for small k .

Example 2.34. (Bases for $C_{k-1}^{(k)}$ and $H_{k-1}(C_*^{(k)})$ for $k = 2, 3, 4$). When $k = 2, 3$, or 4, Theorems 2.38 and 2.43 give the following \mathbb{Z} -bases for the chain group $C_{k-1}^{(k)}$, and the top homology group $H_{k-1}(C_*^{(k)})$. (Here we have taken the graded lexicographical ordering on the set B of Theorem 2.38).

The \mathbb{Z} -basis for the rank-2 group $C_1^{(2)}$ is

$$[1, 2], \quad 12,$$

and $H_1(C_*^{(2)})$ is the rank-one subgroup spanned by $[1, 2] = 12 + 21$. This is the trivial S_2 -representation.

The basis for $C_2^{(3)}$ is

$$[[1, 2], 3], [[1, 3], 2], \quad 1[2, 3], 2[1, 3], 3[1, 2], \quad 123,$$

and $H_2(C_*^{(3)})$ is the rank-two subgroup spanned by

$$[[1, 2], 3] = 123 + 213 - 312 - 321, \quad [[1, 3], 2] = 132 + 312 - 213 - 231$$

isomorphic to the standard S_3 -representation.

The basis for $C_3^{(4)}$ is

$$\begin{aligned} &[[[1, 2], 3], 4], [[[1, 2], 4], 3], [[[1, 3], 2], 4], [[[1, 3], 4], 2], [[[1, 4], 2], 3], [[[1, 4], 3], 2], \\ &[1, 2][3, 4], [1, 3][2, 4], [1, 4][2, 3], \\ &1[[2, 3], 4], 1[[2, 4], 3], 2[[1, 3], 4], 2[[1, 4], 3], 3[[1, 2], 4], 3[[1, 4], 2], 4[[1, 2], 3], 4[[1, 3], 2], \\ &12[3, 4], 13[2, 4], 14[2, 3], 23[1, 4], 24[1, 3], 34[1, 2], \\ &1234. \end{aligned}$$

The top homology group $H_3(C_*^{(4)})$ is the rank-nine free abelian group on the elements

$$\begin{aligned} & [[[1, 2], 3], 4], [[[1, 2], 4], 3], [[[1, 3], 2], 4], [[[1, 3], 4], 2], [[[1, 4], 2], 3], [[[1, 4], 3], 2], \\ & [1, 2][3, 4], [1, 3][2, 4], [1, 4][2, 3], \end{aligned}$$

In general, the homology group will consist of all the basis elements that consist of a product of brackets, that is, the basis elements that contain no singleton factors.

We now introduce notation for the image of the free Lie superalgebra in the free group $C_{k-1}^{(k)}$ on injective words of length k , for $k \geq 2$.

Definition 2.35. For a finite set S with $|S| \geq 2$, let \mathcal{L}_S denote the subset of homogeneous degree- $|S|$ Lie polynomials whose terms are all injective words in S . We write \mathcal{L}_k when $S = [k]$. It is spanned by $(k-1)$ -fold iterated brackets such that each letter in $[k]$ appears exactly once. We define $\mathcal{L}_S = 0$ if S has one or zero elements.

For example, $\mathcal{L}_1 = 0$ by convention. \mathcal{L}_2 is the rank-1 abelian group with basis $[1, 2] = 12 + 21$, and \mathcal{L}_3 is the rank-2 abelian group spanned by the elements

$$[1, [2, 3]] = 123 + 132 - 231 - 321, \quad [2, [1, 3]] = 213 + 231 - 132 - 312, \quad [3, [1, 2]] = 312 + 321 - 123 - 213,$$

which (by the Jacobi identity) sum to zero. As symmetric group representations, \mathcal{L}_2 is the trivial S_2 -representation and \mathcal{L}_3 is the standard S_3 -representation. We give a basis for \mathcal{L}_k using a graded-commutative variation on an argument appearing in Reutenauer [Reu93, Section 5.6.2].

Theorem 2.36 (Compare to Reutenauer [Reu93, Section 5.6.2]). *The abelian group \mathcal{L}_k is free of rank $(k-1)!$ with a \mathbb{Z} -basis all elements of the form*

$$[[[\cdots [1, a_2], a_3], \dots], a_{k-1}], a_k] \quad \text{for any ordering } (a_2, a_3, \dots, a_k) \text{ of the set } \{2, 3, \dots, k\}.$$

More generally, for $S \subseteq [k]$, we define the Reutenauer basis for \mathcal{L}_S to be the $(|S|-1)!$ elements above with the letter 1 replaced by the smallest element of S under the natural ordering on $[k]$.

Proof. As in Reutenauer's proof, we may inductively apply the antisymmetry and Jacobi relations

$$[1, [P, Q]] = (-1)^{|Q||P|+1}[[1, Q], P] + [[1, P], Q]$$

to write any element in \mathcal{L}_k as a linear combination of these generators. The generators must be linearly independent over \mathbb{Z} , since $[[[\cdots [1, a_2], a_3], \dots], a_{k-1}], a_k]$ is the only Lie polynomial in the list whose expansion includes the word $1a_2a_3 \dots a_k$. We note that this last observation also implies that these elements span a direct summand of $C_{k-1}^{(k)}$, and not a higher-index subgroup of a direct summand. \square

Corollary 2.37. *The exponential generating function for the sequence $\ell_k := \text{rank}(\mathcal{L}_k)$ is*

$$\begin{aligned} L(x) &= -\log(1-x) - x \\ &= (1!) \frac{x^2}{2!} + (2!) \frac{x^3}{3!} + (3!) \frac{x^4}{4!} + (4!) \frac{x^5}{5!} + \cdots \end{aligned}$$

In the spirit of the PBW theorem, we will now construct a new basis for the free \mathbb{Z} -module $C_{k-1}^{(k)}$ using the bases defined in Theorem 2.36. Our eventual goal is to prove that a certain subset of this basis spans the top homology group of the complex of injective words.

Theorem 2.38. *Fix a finite set $[k]$ with $k \geq 2$. For each subset $S \subseteq [k]$ with $|S| \geq 2$, let B_S be the basis of \mathcal{L}_S of Theorem 2.36. For each singleton subset $S = \{a\} \subset [k]$, let $B_S = \{a\}$. Put a total order \leq on $B = \sqcup_{S \subseteq [k]} B_S$. Then the set Π of polynomials of the form*

$$P_1 P_2 \cdots P_m \quad \text{such that } [k] = S_1 \sqcup S_2 \sqcup \cdots \sqcup S_m, P_i \in B_{S_i}, \text{ and } P_1 < P_2 < \cdots < P_m \in B$$

is a \mathbb{Z} -basis for $C_{k-1}^{(k)}$.

Proof. The set Π is a subset of the basis given in Ross's superalgebra analogue of the PBW (Theorem 2.33); the elements of Π are linearly independent over $\mathbb{Z}[\frac{1}{2}]$ and therefore over \mathbb{Z} . We must show that they span $C_{k-1}^{(k)}$. Assume without loss of generality that $1 < 2 < \dots < k$ in our total order on B . Observe that the one element of Π associated with the decomposition $[k] = \{1\} \sqcup \{2\} \sqcup \dots \sqcup \{k\}$ is the single word $P = P_1 P_2 \dots P_k = 123 \dots k$. We wish to show all permutations of this word are also contained in the span of Π . We proceed by induction.

Let $\Pi_m \subseteq \Pi$ be the subset of polynomials in Π associated to a decomposition $[k] = S_1 \sqcup S_2 \sqcup \dots \sqcup S_q$ with $q \leq m$. We prove by induction on m that elements in the subset Π_m span the space of all products of elements of B (in any order) with m or fewer factors. This is trivial when $m = 1$; suppose $m > 1$. Observe that, given a polynomial $P = P_1 P_2 \dots P_m \in \Pi$ and a transposition $(i \ i+1) \in S_m$, we have:

$$(P_1 P_2 \dots P_{i+1} P_i \dots P_m) = (-1)^{|P_i||P_{i+1}|} ((P_1 P_2 \dots P_i P_{i+1} \dots P_m) - (P_1 P_2 \dots [P_i, P_{i+1}] \dots P_m)). \quad (1)$$

We may re-express $[P_i, P_{i+1}]$ as a linear combination of Reutenauer basis elements for $\mathcal{L}_{S_i \cup S_{i+1}}$, and by induction $(P_1 P_2 \dots [P_i, P_{i+1}] \dots P_m)$ is in the span of polynomials in Π_{m-1} . Since transpositions of the form $(i \ i+1)$ generate S_m , this implies that all S_m -permutations of the factors of $P = P_1 P_2 \dots P_m$ are in the span of Π_m , which concludes our induction. In particular, when $m = k$ all permutations of our word $P = 123 \dots k$ of length k are contained in the span of $\Pi_k = \Pi$, so $C_{k-1}^{(k)}$ is contained in the span of Π as claimed. \square

Our next goal is to identify $H_{k-1}(C_*^{(k)}) \subseteq C_{k-1}^{(k)}$. We will show that the top homology group is spanned by a certain polynomials we call \mathcal{L} -products.

Definition 2.39. We call an element P of $C_{k-1}^{(k)}$ an \mathcal{L} -product if it has the following form. For some partition of $[k] = S_1 \sqcup S_2 \sqcup \dots \sqcup S_m$, we can decompose P as a product:

$$P = P_1 P_2 \dots P_m \quad \text{with } P_i \in \mathcal{L}_{S_i}.$$

Note that, in contrast to the elements of the basis Π in Lemma 2.38, \mathcal{L} -products exclude factors P_i that are a single letter. For example, the polynomial

$$[1, 2][3, 4] = (12 + 21)(34 + 43) = (1234 + 1243 + 2134 + 2143)$$

is an \mathcal{L} -product in $C_3^{(4)}$, but

$$[1, [2, 3]]4 = (1(23 + 32) - (23 + 32)1)4 = (1234 + 1324 - 2314 - 3214)$$

is *not* an \mathcal{L} -product. The following proposition shows that all \mathcal{L} -products are in the kernel of the differential \mathbf{D} .

Proposition 2.40. Any \mathcal{L} -product in $C_{k-1}^{(k)}$ is a cycle.

Since the homology group $H_{k-1}(C_*^{(k)})$ is the subgroup of cycles in $C_{k-1}^{(k)}$, we may view elements in the span of the \mathcal{L} -products as homology classes.

Proof of Proposition 2.40. We will verify that elements of \mathcal{L}_k are contained in $\ker(\mathbf{D})$. Since the differential \mathbf{D} satisfies the Leibniz rule on elements of $C_{k-1}^{(k)}$

$$\mathbf{D}(ac) = \mathbf{D}(a)c + (-1)^{|a|} a \mathbf{D}(c),$$

it follows that products of these Lie polynomials are in the kernel of \mathbf{D} . We will proceed by induction on k . When $k = 2$ we have $\mathcal{L}_2 = \mathbb{Z}[1, 2]$ and

$$\mathbf{D}([1, 2]) = \mathbf{D}(12 + 21) = 2 - 1 + 1 - 2 = 0.$$

Now fix k and suppose that any Lie polynomial of degree less than k is mapped to zero by \mathbf{D} . To show that $\mathcal{L}_k \subseteq \ker(\mathbf{D})$, it suffices to check Lie polynomials of the form $[P, a_k]$ of Reutenauer's basis (Theorem 2.36). We have:

$$\begin{aligned}
\mathbf{D}([P, a_k]) &= \mathbf{D}(Pa_k - (-1)^{|P|}a_kP) \\
&= \left(\mathbf{D}(P)a_k + (-1)^{|P|}P\mathbf{D}(a_k)\right) - (-1)^{|P|}\left(\mathbf{D}(a_k)P - a_k\mathbf{D}(P)\right) \\
&= 0 + (-1)^{|P|}P\mathbf{D}(a_k) - (-1)^{|P|}\mathbf{D}(a_k)P + 0 && \text{since } \mathbf{D}P = 0 \text{ by the inductive hypothesis,} \\
&= (-1)^{|P|}(P - P) \\
&= 0.
\end{aligned}$$

Thus the Lie polynomials in \mathcal{L}_k and their products are cycles, as claimed. \square

The next result gives a basis for the subgroup of $C_{k-1}^{(k)}$ spanned by \mathcal{L} -products. Theorem 2.43 will then show us that this subgroup is, in fact, precisely the top homology group H_{k-1} .

Lemma 2.41. *Fix a finite set $[k]$ with $k \geq 2$. As in Theorem 2.38, for each subset $S \subseteq [k]$, let B_S be the basis of \mathcal{L}_S of Theorem 2.36. Put a total order \leq on $B = \cup_{S \subseteq [k], |S| \geq 2} B_S$. The set Π^* of polynomials of the form*

$$P_1 P_2 \cdots P_m \quad \text{such that } [k] = S_1 \sqcup S_2 \sqcup \cdots \sqcup S_m, P_i \in B_{S_i}, \text{ and } P_1 < P_2 < \cdots < P_m \in B$$

form a basis for the subgroup of $C_{k-1}^{(k)}$ spanned by \mathcal{L} -products. Moreover, this subgroup is a direct summand of $C_{k-1}^{(k)}$.

Note that, in contrast to Theorem 2.38, our generating set B excludes all words of length 1.

Proof. Because Π^* is a subset of basis Π for $C_{k-1}^{(k)}$ of Theorem 2.38, the polynomials in Π^* must be linearly independent, and their span must be a direct summand of $C_{k-1}^{(k)}$. Each polynomial in Π^* is an \mathcal{L} -product, so it remains to show that they span. As in the proof of Theorem 2.38, we need to show that any permutation of the factors of an element $P_1 P_2 \cdots P_m$ of Π_* is in the span of Π_* , and we may use the same induction argument from Theorem 2.38. Again let $\Pi_m^* \subseteq \Pi^*$ be the subset of polynomials of Π with at most m factors; we prove by induction that Π_m^* spans the space of \mathcal{L} -products with m or fewer factors. When $m = 1$, the polynomials P_1 are precisely the elements in Reutenauer's basis for \mathcal{L}_k (Theorem 2.36). For $m > 1$, Equation (1) in the proof of Theorem 2.38 again completes the inductive step, which concludes our proof. \square

To prove that the subgroup of the chains $C_{k-1}^{(k)}$ given in Lemma 2.41 is in fact the entire top homology group, we will compare their ranks. We now use an Euler characteristic argument to compute the rank of $H_{k-1}(C_*^{(k)})$.

Proposition 2.42. *The top homology group of the complex of injective words is a free abelian group with rank*

$$\text{rank} H_{k-1}(C_*^{(k)}) = \frac{k!}{0!} - \frac{k!}{1!} + \frac{k!}{2!} - \frac{k!}{3!} + \cdots + (-1)^{k-2} \frac{k!}{(k-2)!} + (-1)^{k-1} \frac{k!}{(k-1)!} + (-1)^k \frac{k!}{k!}$$

The exponential generating function for the ranks of these groups is:

$$H(x) = \frac{e^{-x}}{1-x}.$$

The values $h_k := \text{rank} H_{k-1}(C_*^{(k)})$ are well known to be equal to the number of *derangements* in \mathfrak{S}_k , that is, the number of permutations without fixed points. The reasons for this relationship is made clear by the description of the basis in Lemma 2.41; see Remark 2.45.

Proof of Proposition 2.42. Since the group $C_q^{(k)}$ has rank $[\mathfrak{S}_k : \mathfrak{S}_{k-q-1}] = \frac{k!}{(k-q-1)!}$, the Euler characteristic of the augmented chain complex $C_*^{(k)}$ is:

$$\chi = -\frac{k!}{k!} + \frac{k!}{(k-1)!} - \frac{k!}{(k-2)!} + \dots + (-1)^{k-3} \frac{k!}{2!} + (-1)^{k-2} \frac{k!}{1!} + (-1)^{k-1} \frac{k!}{0!}$$

Farmer's results imply that the homology of the complex $C_*^{(k)}$ is a free abelian group concentrated in degree $(k-1)$; see Theorem 2.18. It follows that its Euler characteristic is $(-1)^{k-1} h_k$, where h_k is the nonzero Betti number $h_k = \text{rank} H_{k-1}(C_*^{(k)})$. Thus,

$$h_k = \frac{k!}{0!} - \frac{k!}{1!} + \frac{k!}{2!} - \dots + (-1)^{k-2} \frac{k!}{(k-2)!} + (-1)^{k-1} \frac{k!}{(k-1)!} + (-1)^k \frac{k!}{k!}$$

By inspection, the sequence $\{h_k\}$ satisfies the relation $h_k = kh_{k-1} + (-1)^k$ for $k \geq 1$. Multiplying through by $\frac{x^k}{k!}$ and summing yields:

$$\sum_{k \geq 1} h_k \frac{x^k}{k!} = \sum_{k \geq 1} kh_{k-1} \frac{x^k}{k!} + \sum_{k \geq 1} (-1)^k \frac{x^k}{k!}.$$

Since $h_0 = 1$, we infer that its exponential generating function $H(x)$ satisfies the relation:

$$\begin{aligned} H(x) - 1 &= \sum_{k \geq 1} h_k \frac{x^k}{k!} \\ &= \sum_{k \geq 1} kh_{k-1} \frac{x^k}{k!} + \sum_{k \geq 1} (-1)^k \frac{x^k}{k!} \\ &= \sum_{k \geq 1} h_{k-1} \frac{x^k}{(k-1)!} + \left(\sum_{k \geq 0} \frac{(-x)^k}{k!} - 1 \right) \\ &= xH(x) + e^{-x} - 1. \end{aligned}$$

Solving for $H(x)$ gives:

$$\begin{aligned} H(x) &= \frac{e^{-x}}{1-x} \\ &= 1 + (0) \frac{x}{1!} + (1) \frac{x^2}{2!} + (2) \frac{x^3}{3!} + (9) \frac{x^4}{4!} + (44) \frac{x^5}{5!} + (265) \frac{x^6}{6!} + (1854) \frac{x^7}{7!} + \dots \quad \square \end{aligned}$$

Theorem 2.43. $H_{k-1}(C_*^{(k)})$ is equal to the subgroup of $C_{k-1}^{(k)}$ spanned by \mathcal{L} -products. It has a basis given in Lemma 2.41.

Proof. Because the subgroup spanned by the \mathcal{L} -products is a direct summand of $C_{k-1}^{(k)}$ by Lemma 2.41, to prove the theorem it is enough to prove that its rank is equal to the rank of $H_{k-1}(C_*^{(k)})$. Recall for $k \geq 2$ the basis given in Lemma 2.41,

$$P_1 P_2 \cdots P_m \quad \text{such that } [k] = S_1 \sqcup S_2 \sqcup \cdots \sqcup S_m, P_i \in B_{S_i}, \text{ and } P_1 < P_2 < \dots < P_m \in B$$

In Theorem 2.36 we saw that $|B_a|$ has order $\ell_a = (a-1)!$. The number of ways to decompose $[k]$ into subsets of orders a_1, a_2, \dots, a_m is $\binom{k}{a_1, a_2, \dots, a_m}$, and the number of products of Reutenauer basis elements for these subsets (where factors can appear in any order) is $\binom{k}{a_1, a_2, \dots, a_m} \ell_{a_1} \ell_{a_2} \cdots \ell_{a_m}$. The number of products with factors in ascending order is $\frac{1}{m!} \binom{k}{a_1, a_2, \dots, a_m} \ell_{a_1} \ell_{a_2} \cdots \ell_{a_m}$. Hence the basis Π^* for the space of \mathcal{L} -products given in Lemma 2.41 has cardinality:

$$\ell_k + \frac{1}{2!} \sum_{a+b=k} \binom{k}{a, b} \ell_a \ell_b + \frac{1}{3!} \sum_{a+b+c=k} \binom{k}{a, b, c} \ell_a \ell_b \ell_c + \frac{1}{4!} \sum_{a+b+c+d=k} \binom{k}{a, b, c, d} \ell_a \ell_b \ell_c \ell_d + \dots$$

This implies that the exponential generating function for the rank of this space is given by exponentiating the generating function $L(x) = -\log(1-x) - x$ for ℓ_k found in Corollary 2.37. But

$$e^{L(x)} = \frac{e^{-x}}{1-x} = H(x),$$

where $H(x)$ is the exponential generating function found in Proposition 2.42, and so we conclude that for $k \geq 1$ the cardinality of the basis Π^* is equal to the rank of $H_{k-1}(C_*^{(k)})$. Hence $H_{k-1}(C_*^{(k)})$ is equal to the subgroup of $C_{k-1}^{(k)}$ spanned by \mathcal{L} -products. \square

Remark 2.44. Theorem 2.43 was proven for homology with integer coefficients, but the statement in fact holds over any commutative unital ring R . Since $H_{k-1}(C_*^{(k)})$ is a split subgroup of $C_{k-1}^{(k)}$, Lemma 2.41 gives an R -basis for the top homology group $H_{k-1}(C_*^{(k)}; R)$ of the chain complex $R \otimes_{\mathbb{Z}} C_*^{(k)}$.

Remark 2.45. We remark that Theorem 2.43 and the basis for \mathcal{T}_k given in Lemma 2.41 make it apparent that the rank of \mathcal{T}_k will be equal to the number of derangements of \mathfrak{S}_k . The Reutenauer basis for L_S , $|S| = k$ of Theorem 2.36 are the $(k-1)!$ elements $\{[[[\cdots[a, a_2], a_3], \dots], a_{k-1}], a_k]\}$ where a denotes the lexicographically first element of S and all permutations of the remaining elements a_i of S appear. Then the map

$$[[[\cdots[a, a_2], a_3], \dots], a_{k-1}], a_k] \longmapsto (a \ a_2 \ a_3 \ \cdots \ a_k)$$

identifies the Reutenauer basis elements with the set of k -cycles on S . Extending this map to the basis in Lemma 2.41 identifies each basis element with a permutation without 1-cycles, written in cycle notation, with cycles ordered lexicographically. We have a naturally defined bijection between our basis for \mathcal{T}_k and the set of derangements on k letters. We note, however, that this bijection is not \mathfrak{S}_k -equivariant.

2.4 Secondary injective word complexes

In this subsection, we define a chain complex called the secondary injective words chain complex of a $\bigwedge(\text{Sym}^2 R)$ -module. This chain complex should be thought of as a central stability complex for $\bigwedge(\text{Sym}^2 R)$ -modules and this complex will appear on the E^2 -page of a certain spectral sequence.

Recall that if $A = \{a, b\}$ is a 2-element set, then $\mathcal{L}_A \cong R$ is the free R -module on the graded Lie bracket $[a, b] = ab + ba$, that is, \mathcal{L}_A is a rank-one trivial \mathfrak{S}_2 -representation.

Definition 2.46. Let \mathcal{V} be a $\bigwedge(\text{Sym}^2 \mathbb{Z})$ -module, S a set of cardinality k . Let

$$\begin{aligned} \text{Inj}_p^2(\mathcal{V})_S &= \text{Ind}_{(\mathfrak{S}_2)^{p+1} \times \mathfrak{S}_{k-2p-2}}^{\mathfrak{S}_k} \mathcal{L}_2^{\boxtimes(p+1)} \boxtimes \mathcal{V}_{k-2p-2} \\ &= \bigoplus_{\substack{S=A_0 \sqcup A_1 \sqcup \cdots \sqcup A_p \sqcup B \\ |A_i|=2, |B|=k-2p-2}} \mathcal{L}_{A_0} \otimes \mathcal{L}_{A_1} \otimes \cdots \otimes \mathcal{L}_{A_p} \otimes \mathcal{V}_B. \end{aligned}$$

These groups assemble to form a semi-simplicial $\bigwedge(\text{Sym}^2 \mathbb{Z})$ -module. The category FIM^+ acts as follows. Consider a morphism

$$F = (f, T_1 \wedge T_2 \wedge \cdots \wedge T_k) \in \text{Hom}_{\text{FIM}^+}(S, T)$$

and a summand

$$\mathcal{L}_{A_0} \otimes \mathcal{L}_{A_1} \otimes \cdots \otimes \mathcal{L}_{A_p} \otimes \mathcal{V}_B \quad \text{of} \quad \text{Inj}_p^2(\mathcal{V})_S.$$

The morphism F maps the tensor factors $\mathcal{L}_{A_0} \otimes \mathcal{L}_{A_1} \otimes \cdots \otimes \mathcal{L}_{A_p}$ to the factors $\mathcal{L}_{f(A_0)} \otimes \mathcal{L}_{f(A_1)} \otimes \cdots \otimes \mathcal{L}_{f(A_p)}$. It acts on \mathcal{V}_B by the FIM^+ morphism

$$(f|_B, T_1 \wedge T_2 \wedge \cdots \wedge T_k) \in \text{Hom}_{\text{FIM}^+}(B, T \setminus \sqcup_i f(A_i)).$$

We define the facemap

$$\begin{aligned} d_i : \text{Inj}_p^2(\mathcal{V})_S &\longrightarrow \text{Inj}_{p-1}^2(\mathcal{V})_S & (i = 0, \dots, p) \\ \mathcal{L}_{A_0} \otimes \mathcal{L}_{A_1} \otimes \cdots \otimes \mathcal{L}_{A_p} \otimes \mathcal{V}_B &\longrightarrow \mathcal{L}_{A_0} \otimes \mathcal{L}_{A_1} \otimes \cdots \otimes \widehat{\mathcal{L}_{A_i}} \otimes \cdots \otimes \mathcal{L}_{A_{p-1}} \otimes \mathcal{V}_{B \sqcup A_i} \end{aligned}$$

as follows: let d_i act by the identity on the tensor factors $\mathcal{L}_{A_0}, \mathcal{L}_{A_1}, \dots, \widehat{\mathcal{L}_{A_i}}, \dots, \mathcal{L}_{A_{p-1}}$, and act on the factor \mathcal{V} by the map $\mathcal{V}_B \rightarrow \mathcal{V}_{B \sqcup A_i}$ induced by the FIM^+ morphism associated to the inclusion $(B \hookrightarrow B \sqcup A_i, A_i)$. Let $\text{Inj}_*^2(\mathcal{V})_S$ denote the chain complex constructed by taking the alternating sum of the face maps as differentials.

Let S be a set of cardinality k . We now define a chain complex closely related to the complex of injective words, called the *oriented complex of injective words* and denoted $\text{Inj}_*^+(S)$. We will show that, as a chain complex, it is isomorphic to the reduced cellular chains on the usual complex of injective words, however it is not equivariantly isomorphic.

Definition 2.47. In degree p , let $\text{Inj}_p^+(S)$ be the free R -module on elements of the form

$$a_0 a_1 a_2 \cdots a_p \otimes a_{p+1} \wedge a_{p+2} \wedge \cdots \wedge a_k$$

with a_i distinct elements of S and the differential $\partial_p^+ : \text{Inj}_p^+(S) \rightarrow \text{Inj}_{p-1}^+(S)$ is given by the (not-alternating) sum of the $(p+1)$ maps

$$\begin{aligned} d_i^+ : \text{Inj}_p^+(S) &\longrightarrow \text{Inj}_{p-1}^+(S) & i = 0, \dots, p \\ a_0 a_1 a_2 \cdots a_p \otimes a_{p+1} \wedge a_{p+2} \wedge \cdots \wedge a_k &\longmapsto a_0 a_1 a_2 \cdots \widehat{a_i} \cdots a_p \otimes a_i \wedge a_{p+1} \wedge a_{p+2} \wedge \cdots \wedge a_k \end{aligned}$$

Proposition 2.48. *The oriented complex of injective words $\text{Inj}_*^+(S)$ on a set S is isomorphic as chain complexes to the reduced cellular chains on the classical injective word complex $\text{Inj}_*(S)$.*

Since the complex of injective words is highly connected (Theorem 2.18), the oriented complex of injective words is highly acyclic.

Corollary 2.49. *The homology groups $H_i(\text{Inj}_*^+(S)) \cong 0$ for $i \leq |S| - 2$.*

Proof of Proposition 2.48. Let $S = \{a, b, c, \dots\}$ be an alphabet on k letters, with a choice of order $a < b < c < \dots$. We will exhibit a chain isomorphism Φ from the complex of injective words Inj_* to the oriented complex of injective words Inj_*^+ . In degree p we define Φ as follows. For the injective word $w = a_0 a_1 \cdots a_p$, fix an order on the (possibly empty) complement $S \setminus \{a_i\} = \{b_1, \dots, b_{k-p-1}\}$, and let ϵ_w denote the sign of the permutation in \mathfrak{S}_k that transforms the word $a_0 a_1 \cdots a_p b_1 b_2 \cdots b_{k-p-1}$ into the chosen order $abcd \cdots$ on S . Then let $\Phi : \text{Inj}_p(S) \longrightarrow \text{Inj}_p^+(S)$ be given by the formula:

$$\Phi(a_0 a_1 \cdots a_p) = \epsilon_w (-1)^{\frac{(p+1)(p)}{2}} a_0 a_1 \cdots a_p \otimes b_1 \wedge b_2 \wedge \cdots \wedge b_{k-p-1}$$

A routine computation shows that this map is a chain isomorphism. \square

Remark 2.50. The equivalence of the chain complexes $\text{Inj}_*(S)$ and $\text{Inj}_*^+(S)$ of Proposition 2.48 does not respect the symmetric group action. Note for example that for $S = \{a, b\}$, the top homology of $\text{Inj}_*(S)$ is the rank-1 free abelian group on the element $(ab + ba)$, which carries a trivial \mathfrak{S}_2 -action, whereas the top homology of $\text{Inj}_*^+(S)$ is the rank-1 free abelian group on the element $(ab - ba)$, the sign representation of \mathfrak{S}_2 . Fortunately this nonequivariant equivalence suffices to establish high acyclicity.

Proposition 2.51. *There is an isomorphism of chain complexes:*

$$\text{Inj}_*^2 \left(M(d)^{\text{FIM}^+} \right)_S = \bigoplus_{\substack{f \in \text{Hom}_{\text{FI}}([d], S) \\ Z \text{ a perfect matching on } S \setminus \text{im}(f)}} \text{Inj}_*^+(Z).$$

Proposition 2.51 and Corollary 2.49 imply that $\text{Inj}_M^2(d)_S^{\text{FIM}^+}$ is highly acyclic.

Corollary 2.52. *The homology groups $H_i(\text{Inj}_M^2(d)_S^{\text{FIM}^+}) \cong 0$ if $i \leq \left(\frac{|S|-d}{2} - 2 \right)$.*

Proof Proposition 2.51. By definition, $M(d)_B^{\text{FIM}^+} = \text{Hom}_{\text{FIM}^+}([d], B)$, and so

$$\text{Inj}_*^2 \left(M(d)^{\text{FIM}^+} \right)_S = \bigoplus_{\substack{[k]=A_0 \sqcup A_1 \sqcup \dots \sqcup A_p \sqcup B \\ |A_i|=2, |B|=k-2p-2}} \mathcal{L}_{A_0} \otimes \mathcal{L}_{A_1} \otimes \dots \otimes \mathcal{L}_{A_p} \otimes \text{Hom}_{\text{FIM}^+}([d], B)$$

with the face map d_i acting on $\text{Inj}_*^2 \left(M(d)^{\text{FIM}^+} \right)_S$ by mapping the term

$$\mathcal{L}_{A_0} \otimes \mathcal{L}_{A_1} \otimes \dots \otimes \mathcal{L}_{A_p} \otimes \left(f : [d] \rightarrow B, B_1 \wedge B_2 \wedge \dots \wedge B_{\frac{k-2p-2-d}{2}} \right)$$

to the term

$$\mathcal{L}_{A_0} \otimes \mathcal{L}_{A_1} \otimes \dots \otimes \widehat{\mathcal{L}_{A_i}} \otimes \dots \otimes \mathcal{L}_{A_{p-1}} \otimes \left(f : [d] \rightarrow B, A_i \wedge B_1 \wedge B_2 \wedge \dots \wedge B_{\frac{k-2p-2-d}{2}} \right).$$

Hence we can realize $\text{Inj}_*^2 \left(M(d)^{\text{FIM}^+} \right)_S$ as

$$\begin{aligned} \text{Inj}_*^2 \left(M(d)^{\text{FIM}^+} \right)_S &= \bigoplus_{\substack{[k]=A_0 \sqcup A_1 \sqcup \dots \sqcup A_p \sqcup B, \quad |A_i|=2, \quad |B|=k-2p-2 \\ f:[d] \hookrightarrow B, \quad B_1 \wedge B_2 \wedge \dots \wedge B_{(k-2p-2-d)/2}, \quad |B_i|=2 \\ B=B_1 \sqcup B_2 \sqcup \dots \sqcup B_{(k-2p-2-d)/2} \sqcup \text{im}(f)}} \mathcal{L}_{A_0} \otimes \mathcal{L}_{A_1} \otimes \dots \otimes \mathcal{L}_{A_p} \\ &= \bigoplus_{\substack{f:[d] \hookrightarrow [k] \\ \text{oriented matching of order } k-2p-2-d \text{ on } [k] \setminus \text{im}(f), \\ \text{ordered matching } (A_0, \dots, A_p) \text{ on complement of } \text{im}(f) \text{ and oriented matching}}} \mathcal{L}_{A_0} \otimes \mathcal{L}_{A_1} \otimes \dots \otimes \mathcal{L}_{A_p} \\ &= \bigoplus_{\substack{f:[d] \hookrightarrow [k] \\ \text{ordered matching } (A_0, \dots, A_p) \\ \text{disjoint oriented matching } A_{p+1} \wedge \dots \wedge A_{\frac{k-d}{2}}}} \mathcal{L}_{A_0} \otimes \mathcal{L}_{A_1} \otimes \dots \otimes \mathcal{L}_{A_p} \otimes \left(\mathcal{L}_{A_{p+1}} \wedge \dots \wedge \mathcal{L}_{A_{\frac{k-d}{2}}} \right). \end{aligned}$$

Here, the face map d_i maps the summand

$$\mathcal{L}_{A_0} \otimes \mathcal{L}_{A_1} \otimes \dots \otimes \mathcal{L}_{A_p} \otimes \left(\mathcal{L}_{A_{p+1}} \wedge \dots \wedge \mathcal{L}_{A_{\frac{k-d}{2}}} \right)$$

to the summand

$$\mathcal{L}_{A_0} \otimes \mathcal{L}_{A_1} \otimes \dots \otimes \widehat{\mathcal{L}_{A_i}} \otimes \dots \otimes \mathcal{L}_{A_{p-1}} \otimes \left(\mathcal{L}_{A_i} \wedge \mathcal{L}_{A_{p+1}} \wedge \dots \wedge \mathcal{L}_{A_{\frac{k-d}{2}}} \right).$$

Thus we can identify $\text{Inj}_*^2 \left(M(d)^{\text{FIM}^+} \right)_S$ with the complex

$$\text{Inj}_*^2 \left(M(d)^{\text{FIM}^+} \right)_S = \bigoplus_{\substack{f:[d] \hookrightarrow [k] \\ Z \text{ a perfect matching on } S \setminus \text{im}(f)}} \text{Inj}_*^+(Z)$$

as desired. \square

The following is a corollary of Theorem 2.14.

Corollary 2.53. *Let R be field of characteristic zero and let \mathcal{V} be a finitely generated $\bigwedge(\text{Sym}^2 R)$ -module. There are integers d_i, e_i, m_{ij} and maps making the following an exact sequence of $\bigwedge(\text{Sym}^2 R)$ -modules:*

$$\dots \rightarrow \bigoplus_{j=d_1}^{e_1} (M(j)^{\text{FIM}^+})^{m_{1j}} \rightarrow \bigoplus_{j=d_0}^{e_0} (M(j)^{\text{FIM}^+})^{m_{0j}} \rightarrow \mathcal{V} \rightarrow 0$$

Proof. Suppose for the purposes of induction that we have constructed the first k stages of a resolution by modules of the form $\bigoplus_{j=d_i}^{e_i} (M(j)^{\text{FIM}^+})^{m_{ij}}$. The kernels of the last map is a submodule of a finitely generated FIM^+ -module. Hence it is finitely generated and so there exists a surjection onto it from an FIM^+ -module of this form. Using this map, we construct the next term in the sequence. \square

Proposition 2.54. *Let R be a field of characteristic zero and let \mathcal{V} be a finitely generated $\bigwedge(\text{Sym}^2 R)$ -module. For each p , there is a number $N_p^\mathcal{V}$ such that if $|S| > N_p^\mathcal{V}$, the homology group $H_p(\text{Inj}_*^2(\mathcal{V}))_S$ vanishes.*

Proof. Consider a free resolution of \mathcal{V} as in Corollary 2.53:

$$\dots \rightarrow \bigoplus_{j=d_1}^{e_1} (M(j)^{\text{FIM}^+})^{m_{1j}} \rightarrow \bigoplus_{j=d_0}^{e_0} (M(j)^{\text{FIM}^+})^{m_{0j}} \rightarrow \mathcal{V} \rightarrow 0.$$

Let C_* be the above chain complex with \mathcal{V} replaced with 0. Note that the functor $\mathcal{W} \mapsto \text{Inj}_p^2(\mathcal{W})$ is exact for all p . Consider the double complex spectral sequences associated to the double complex $\text{Inj}_*^2(C_*)$. One spectral sequence has:

$$E_{p,q}^2 = H_p(\text{Inj}_*(H_q(C_*))).$$

Since $H_q(C_*)$ vanishes for $q > 0$, this spectral sequence collapses on the second page. Since $H_0(C_*) = \mathcal{V}$, this spectral sequence converges to $H_p(\text{Inj}_*^2(\mathcal{V}))$. The other spectral sequence has:

$${}'E_{p,q}^1 = H_p(\text{Inj}_*^2(C_q)).$$

Since

$$C_q = \text{Inj}_*^2 \left(\bigoplus_{j=d_q}^{e_q} (M(j)^{\text{FIM}^+})^{m_{qj}} \right),$$

Corollary 2.52 implies that ${}'E_{p,q}^1(S)$ vanishes in range increasing with the size of S . Thus, this spectral sequence converges to zero in a range increasing with the size of S . This implies that $H_p(\text{Inj}_*^2(\mathcal{V}))_S \cong 0$ for S sufficiently large compared with p . \square

The following corollary shows that Theorem 1.4 implies Corollary 1.5.

Corollary 2.55. *Let R be a field of characteristic zero and \mathcal{V} a finitely generated $\bigwedge(\text{Sym}^2 R)$ -module. For k sufficiently large, \mathcal{V}_k is isomorphic to the coequalizer of two natural maps:*

$$\text{Ind}_{S_{k-4} \times S_2 \times S_2}^{S_k} \mathcal{V}_{k-4} \rightrightarrows \text{Ind}_{S_{k-2} \times S_2}^{S_k} \mathcal{V}_{k-2}$$

Proof. This condition involving coequalizers is exactly the statement that

$$H_0(\text{Inj}_*^2(\mathcal{V}))_k \cong H_{-1}(\text{Inj}_*^2(\mathcal{V}))_k \cong 0.$$

This is true for large k by Proposition 2.54. \square

3 Configuration spaces of noncompact manifolds

In this section, we apply the tools of the previous section to prove secondary representation stability. We begin by recalling the definition of configuration spaces and their stabilization maps in Section 3.1. Then, in Section 3.2, we define the *arc resolution* and an associated spectral sequence, which we use to give a new proof of representation stability for configuration spaces. In Section 3.3, we compute the differentials in this spectral sequence, and use this calculation to prove secondary representation stability for configuration spaces of surfaces in Section 3.4, as well as an improved range for representation stability for configuration spaces of high dimensional manifolds in Section 3.5. In Section 3.6, we give some computations for specific manifolds, and some conjectures.

3.1 Stabilization maps and homology operations

In this subsection, we define stabilization maps and an FI-module structure on the homology of the configuration spaces of a noncompact manifold. Throughout the section M will always denote a connected manifold of dimension $n \geq 2$. Manifolds in this paper are assumed to be without boundary, unless otherwise stated. We will frequently refer to elements of M as ‘particles’ in an effort to avoid the ambiguity that comes with phrases like “points in a configuration space.” For simplicity, we will assume that M is a smooth manifold, although all results are true for general topological manifolds. See Remark 3.12 for a discussion of the necessary modifications needed to address non-smoothable manifolds.

Definition 3.1. For smooth manifolds M and N (possibly with boundary), let $\text{Emb}(N, M)$ denote the space of smooth embeddings of N into M , topologized with the C^∞ topology.

Definition 3.2. Given a finite set S , let $F_S(M) = \text{Emb}(S, M)$. We write $F_k(M)$ for $F_{[k]}(M)$. Let $C_k(M)$ denote the quotient of $F_k(M)$ by the action of $\mathfrak{S}_k = \text{Aut}([k])$. The space

$$F_k(M) \cong \{(m_1, \dots, m_k) \in M^k \mid m_i \neq m_j \text{ for } i \neq j\}$$

is the configuration space of k ordered particles in M , and the space $C_k(M)$ is the configuration space of k unordered particles in M .

Given an embedding of manifolds $N \sqcup L \rightarrow M$ and sets S and T , we get a map of spaces

$$F_S(N) \times F_T(L) \rightarrow F_{S \sqcup T}(M).$$

If M is not compact, there exists an embedding $e : \mathbb{R}^n \sqcup M \hookrightarrow M$ with $e|_M$ isotopic to the identity, as described in Section 1 (see Figure 1). We fix such an embedding for the duration of this paper. With this embedding we define the following maps on the homology of configuration spaces.

Definition 3.3. Let M be a noncompact manifold. Given a class $\alpha \in H_i(F_S(\mathbb{R}^n))$, let

$$t_\alpha : H_*(F_T(M)) \longrightarrow H_{*+i}(F_{T \sqcup S}(M))$$

be the map on homology induced by the embedding $e : \mathbb{R}^n \sqcup M \hookrightarrow M$.

The sequence of \mathfrak{S}_k -representations $H_i(F_k(M))$ assemble to form an FI-module as follows. For a set P , let $[P]$ be the class of a point in $H_0(F_P(\mathbb{R}^n))$. Let $f : S \rightarrow T$ be an injective map of finite sets. The FI-module structure on $H_i(F(M))$ is defined so that the map f is sent to the composition of the map induced by the homeomorphism $F_S(M) \rightarrow F_{f(S)}(M)$ and $t_{[T \setminus f(S)]}$. See Figure 4 for an illustration. This FI-module structure on homology arises from a homotopy-FI-space structure on the functor $S \rightarrow F_S(M)$.

One can also define a co-FI-space structure $S \mapsto F_S(M)$ on configuration spaces as follows. View $F_S(M)$ as the spaces of embeddings $\text{Emb}(S, M)$ and let injections act by precomposition, as in Figure 11. The induced co-FI-module structure on $H_i(F(M))$ is compatible with the FI-module structure in

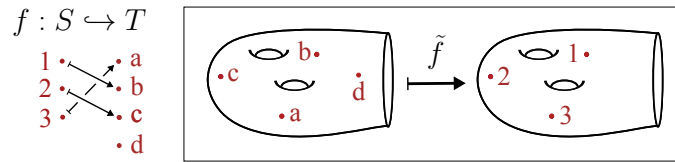


Figure 11: The co-FI-space structure on $F(M)$.

such a way as to give $H_i(F(M))$ the structure of an FI-module. Church, Ellenberg, and Farb describe this structure in detail [CEF15, Section 6].

Generalizing the construction of the stabilization map, for manifolds N, L, M there is a natural map:

$$\text{Emb}(N \sqcup L, M) \times F_S(N) \times F_T(L) \rightarrow F_{S \sqcup T}(M).$$

Define a map

$$\theta : S^{n-1} \rightarrow \text{Emb}(\mathbb{R}^n \sqcup \mathbb{R}^n, \mathbb{R}^n)$$

as follows: Let $\tau : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a map which induces a homeomorphism between \mathbb{R}^n and the open unit ball around the origin. View S^{n-1} as the unit vectors in \mathbb{R}^n and let $\theta(\vec{v}) : \mathbb{R}^n \sqcup \mathbb{R}^n \rightarrow \mathbb{R}^n$ be the function mapping \vec{x} in the first copy of \mathbb{R}^n to $\theta(\vec{x}) + \vec{v}$ and mapping \vec{x} in the second copy of \mathbb{R}^n to $\theta(\vec{x}) - \vec{v}$. By restricting to the class of a point in $H_0(S^{n-1})$, this induces a product on the homology of ordered configuration space of \mathbb{R}^n :

$$\bullet : H_i(F_S(\mathbb{R}^n)) \otimes H_j(F_T(\mathbb{R}^n)) \rightarrow H_{i+j}(F_{S \sqcup T}(\mathbb{R}^n)).$$

By restricting to the fundamental class of S^{n-1} , this induces a bracket:

$$\psi^n : H_i(F_S(\mathbb{R}^n)) \otimes H_j(F_T(\mathbb{R}^n)) \rightarrow H_{i+j+n-1}(F_{S \sqcup T}(\mathbb{R}^n)).$$

This can be thought of as a version of the Browder operation for E_n -left modules. See Markl, Shnider, and Stasheff [MSS07, Definition 3.26] for the definition of left modules over an operad, May [May72, Definition 4.1.] for the definition of E_n -operads, and Browder [Bro60, Page 351] for the definition of Browder operations in the context of higher H -spaces. The operations \bullet and ψ^n are illustrated in Figure 12.

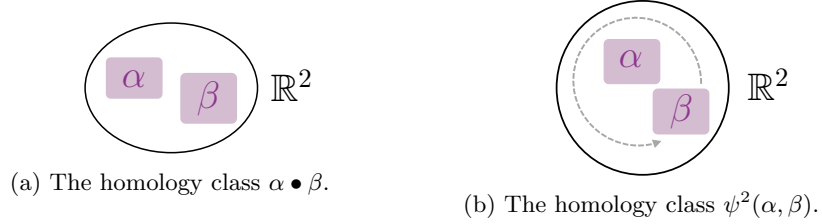


Figure 12: Chains representing the homology operations on $H_*(F(\mathbb{R}^n))$.

In this paper, we are primarily interested in the operation ψ^2 , which we simply call ψ . We define ψ in dimension $n \geq 2$ as follows. Let

$$\theta' : S^1 \rightarrow \text{Emb}(\mathbb{R}^n \sqcup \mathbb{R}^n, \mathbb{R}^n)$$

be the restriction of θ to an equatorial circle. The fundamental class of S^1 induces a map:

$$\psi : H_i(F_S(\mathbb{R}^n)) \otimes H_j(F_T(\mathbb{R}^n)) \rightarrow H_{i+j+1}(F_{S \sqcup T}(\mathbb{R}^n)).$$

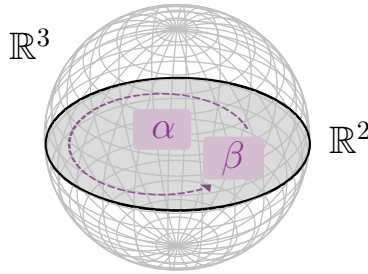


Figure 13: The homology class $\psi(\alpha, \beta) \in H_*(F(\mathbb{R}^3))$.

Figure 13 shows the map ψ on $H_*(F(\mathbb{R}^3))$. Although for $n > 2$, ψ is the zero homology operation, it is nonzero at the chain level and this will be relevant in Section 3.3. Given a set $S = \{s\}$, let s denote the class of a point in $H_0(F_S(\mathbb{R}^n))$. Figure 14 shows $\psi(1, 2) \in H_1(F_2(\mathbb{R}^n))$.

Cohen described the algebraic structure on $H_*(F_k(\mathbb{R}^n))$ imposed by the operations ψ^n and \bullet : the groups $H_*(F_k(\mathbb{R}^n))$ assemble to form the n -Poisson operad. Cohen denotes the Browder operations by λ_{n-1} and details the relations they satisfy in [CLM76, Chapter III Theorem 1.2].



Figure 14: A chain representing $\psi(1, 2)$.

Theorem 3.4 (Cohen [CLM76, Chapter III]). *Fix $n \geq 2$. The product \bullet is an associative and graded commutative product, and the Browder operation ψ^n is a graded Lie bracket of degree $(n-1)$, which together satisfy the Gerstenhaber relations. Specifically, these operations satisfy the following identities. Let $|\cdot|$ denote the degree of a homology class.*

(Degrees of \bullet and ψ^n)

$$|\alpha \bullet \beta| = |\alpha| + |\beta| \quad |\psi^n(\alpha, \beta)| = |\alpha| + |\beta| + (n-1)$$

(Graded commutativity for \bullet)

$$\alpha \bullet \beta = (-1)^{|\alpha||\beta|} \beta \bullet \alpha$$

(Graded antisymmetry law for ψ^n)

$$\psi^n(\alpha, \beta) = -(-1)^{|\alpha||\beta| + (n-1)(|\alpha| + |\beta| + 1)} \psi^n(\beta, \alpha)$$

(Graded Jacobi identity for ψ^n)

$$\begin{aligned} & (-1)^{(|\alpha| + n - 1)(|\gamma| + n - 1)} \psi^n(\alpha, \psi^n(\beta, \gamma)) + (-1)^{(|\beta| + n - 1)(|\alpha| + n - 1)} \psi^n(\beta, \psi^n(\gamma, \alpha)) \\ & + (-1)^{(|\gamma| + n - 1)(|\beta| + n - 1)} \psi^n(\gamma, \psi^n(\alpha, \beta)) = 0 \end{aligned}$$

(The Browder operation ψ^n is a derivation of the product \bullet in each variable)

$$\psi^n(\alpha, \beta \bullet \gamma) = \psi^n(\alpha, \beta) \bullet \gamma + (-1)^{(|\alpha| + n - 1)|\beta|} \beta \bullet \psi^n(\alpha, \gamma)$$

Remark 3.5. When we say that \bullet is a commutative product, we do not mean that $\bigoplus_{i,k} H_i(F_k(\mathbb{R}^n))$ is a graded commutative algebra. Instead, we mean that $\{\bigoplus_{i,k} H_i(F_k(\mathbb{R}^n))\}_{k=0}^{k=\infty}$ is a graded left-module over the commutative operad. Similarly, ψ^n makes an appropriate shift of $\{\bigoplus_{i,k} H_i(F_k(\mathbb{R}^n))\}_{k=0}^{k=\infty}$ into a graded left-module over the Lie operad. See for example Markl–Shnider–Stasheff [MSS07] for the definition of these operads.

3.2 The arc resolution and representation stability

We now recall two related semi-simplicial spaces. One was used by Kupers and Miller [KM14, Appendix] to give a new proof of homological stability for unordered configuration spaces. We will use the second to give a new proof of representation stability for ordered configuration spaces.

Definition 3.6. Let M be the interior of a (not necessarily compact) smooth manifold \overline{M} with boundary ∂M . Assume ∂M is nonempty and fix an embedding $\gamma : [0, 1] \rightarrow \partial M$. Such a manifold \overline{M} exists whenever M is noncompact (see for example Miller–Palmer [MP15, Section 3]).

Let

$$\text{Arc}_j(F_k(M)) \subset F_k(M) \times \text{Emb}(\sqcup_{j+1}[0, 1], \overline{M})$$

be the subspace of particles and arcs $(x_1, \dots, x_k; \alpha_0, \dots, \alpha_j)$ satisfying the following conditions:

- $\alpha_i(0) \in \gamma([0, 1])$
- $\alpha_i(t) \notin \partial M \cup \{x_1, \dots, x_k\}$ for $t \in (0, 1)$
- $\alpha_i(1) \in \{x_1, \dots, x_k\}$
- $\gamma^{-1}(\alpha_{j_1}(0)) > \gamma^{-1}(\alpha_{j_2}(0))$ whenever $j_1 > j_2$.

Let $\text{Arc}_j(C_k(M))$ denote the quotient of $\text{Arc}_j(F_k(M))$ by the action of \mathfrak{S}_k , as in Figure 15.

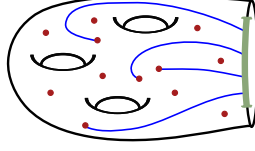


Figure 15: An element of $\text{Arc}_3(C_{12}(M))$.

As j varies, the spaces $\text{Arc}_j(F_k(M))$ assemble to form an augmented semi-simplicial space. The i th face map

$$d_i : \text{Arc}_j(F_k(M)) \rightarrow \text{Arc}_{j-1}(F_k(M))$$

is given by forgetting the i th arc α_i . The space $\text{Arc}_{-1}(F_k(M))$ is homeomorphic to $F_k(M)$, and so the augmentation map induces a map $||\text{Arc}_\bullet(F_k(M))|| \rightarrow F_k(M)$. Similarly $\text{Arc}_j(C_k(M))$ assemble to form an augmented semi-simplicial space and

$$\text{Arc}_{-1}(C_k(M)) \cong C_k(M).$$

We call the two augmented semi-simplicial spaces $\text{Arc}_\bullet(F_k(M))$ and $\text{Arc}_\bullet(C_k(M))$ the *ordered* and *unordered arc resolutions*, respectively.

Building on the work of Hatcher and Wahl [HW10] and a lecture of Randal-Williams, Kupers–Miller [KM14, Appendix] proved the following.

Theorem 3.7 (Kupers–Miller [KM14, Appendix]). *Let M be a noncompact connected manifold of dimension at least two. The map $||\text{Arc}_\bullet(C_k(M))|| \rightarrow C_k(M)$ is $(k-1)$ -connected.*

This implies the same connectivity for the arc resolution of ordered configuration spaces.

Proposition 3.8. *Let M be a noncompact connected manifold of dimension at least two. The map $||\text{Arc}_\bullet(F_k(M))|| \rightarrow F_k(M)$ is $(k-1)$ -connected.*

Proof. Since the map $||\text{Arc}_\bullet(C_k(M))|| \rightarrow C_k(M)$ is $(k-1)$ -connected, its homotopy fibers (the standard path space construction) are $(k-2)$ -connected. The quotient map $F_k(M) \rightarrow C_k(M)$ induces homeomorphisms between the homotopy fibers of $||\text{Arc}_\bullet(F_k(M))|| \rightarrow F_k(M)$ and the homotopy fibers of $||\text{Arc}_\bullet(C_k(M))|| \rightarrow C_k(M)$. Thus the homotopy fibers of $||\text{Arc}_\bullet(F_k(M))|| \rightarrow F_k(M)$ are $(k-2)$ -connected and so the map $||\text{Arc}_\bullet(F_k(M))|| \rightarrow F_k(M)$ is $(k-1)$ -connected as well. \square

If M is connected and of dimension at least two, then the connected components of $\text{Arc}_j(F_k(M))$ are determined by which arc connects to which particle. For example, a connected component could be specified by saying that α_0 connects to x_3 , that α_1 connects to x_8 , and so forth. Thus $\text{Arc}_j(F_k(M))$ has $k!/(k-j-1)!$ connected components. Kupers and Miller [KM14, Appendix] showed that $\text{Arc}_j(C_k(M))$ is homotopy equivalent to $C_{k-j-1}(M)$, and their result implies that each connected component of $\text{Arc}_j(F_k(M))$ is homotopy equivalent to $F_{k-j-1}(M)$. The face maps of the unordered arc resolution are homotopic to the stabilization maps for unordered configuration spaces [KM14, Appendix]. It follows that the face map on $\text{Arc}_j(F_k(M))$ that forgets the arc attached to the particle x_i has the effect of stabilizing by a particle labeled by i .

An augmented semi-simplicial space A_\bullet gives rise to a homology spectral sequence; see for example Randal-Williams [RW13, Section 2.3]. This spectral sequence satisfies

$$E_{p,q}^1 = H_p(A_q) \implies H_{p+q+1}(A_{-1}, ||A_\bullet||),$$

and the differentials on the E^1 -page are given by the alternating sum of the face maps.

Definition 3.9. We call the spectral sequence associated to an (augmented) semi-simplicial space A_\bullet the *(augmented) geometric realization spectral sequence*. We call the augmented geometric realization spectral sequence for the ordered arc resolution the *arc resolution spectral sequence*.

Proposition 3.10. *Let M be a noncompact connected manifold of dimension at least two. The arc resolution spectral sequence satisfies:*

$$E_{p,q}^1(S) \cong \text{Inj}_p(H_q(F_S(M))) \quad \text{for } q \geq 0 \text{ and } p \geq -1.$$

It converges to:

$$H_{p+q+1}(F_S(M), ||\text{Arc}_\bullet(F_S(M))||).$$

Figure 16: $E_{p,q}^1(S) = H_q(\text{Arc}_p(F_S(M))) \cong \text{Inj}_p(H_q(F(M)))_S$.

For $|S| = k$, the E^2 -page satisfies

$$\begin{aligned} E_{p,q}^2(S) &\cong \bigoplus_{\substack{S=P \sqcup Q, \\ |P|=p+1}} H_p(\text{Inj}_*(P)) \otimes H_0^{\text{FI}}(H_q(F(M)))_Q \\ &\cong \text{Ind}_{\mathfrak{S}_{p+1} \times \mathfrak{S}_{k-p-1}}^{\mathfrak{S}_k} H_p(\text{Inj}_*(p+1)) \boxtimes H_0^{\text{FI}}(H_q(F(M)))_{k-p-1}. \end{aligned}$$

Figure 17: $E_{p,q}^2(6) \cong \text{Ind}_{\mathfrak{S}_{p+1} \times \mathfrak{S}_{6-p-1}}^{\mathfrak{S}_6} \mathcal{T}_p \boxtimes H_0^{\text{FI}}(H_q(F(M)))_{6-p-1}$.

In particular, the leftmost E^2 column $p = -1$ are the FI-homology groups

$$E_{-1,q}^2(S) \cong H_0^{\text{FI}}(H_q(F(M)))_S$$

and the bottom E^2 row $q = 0$ are the reduced homology groups of the complex of injective words

$$E_{p,0}^2(S) \cong \tilde{H}_p(\text{Inj}_*(S))$$

which vanish except at $p = k - 1$.

The E_1 -page and E_2 -page of the arc resolution spectral sequence are shown in Figures 16 and 17.

Proof of Proposition 3.10. By definition, the arc resolution spectral sequence satisfies

$$E_{p,q}^1(S) = H_q(\text{Arc}_p(F_S(M))) \quad \text{for } q \geq 0 \text{ and } p \geq -1.$$

Since $F_S(M)$ is the space of (-1) -simplices, the spectral sequence converges to $H_{p+q+1}(F_S(M), ||\text{Arc}_\bullet(M)||)$ as claimed.

By the above remarks on the structure of the space $\text{Arc}_p(F_S(M))$, the E^1 -page satisfies

$$E_{p,q}^1(S) = H_q(\text{Arc}_p(F_S(M))) \cong \bigoplus_{f: \{0,1,\dots,p\} \hookrightarrow S} H_q(F_{S \setminus \text{im}(f)}(M))$$

and has d^1 differentials induced by the alternating sum of face maps on $\text{Arc}_\bullet(F_S(M))$ which are homotopic to stabilization maps. Hence each row of the E^1 -page is precisely the twisted complex of injective words

$$E_{p,q}^1(S) \cong \text{Inj}_p\left(H_q(F(M))\right)_S$$

of Definition 2.19. It follows that

$$E_{p,q}^2(S) \cong H_p\left(\text{Inj}_*(H_q(F(M)))\right)_S.$$

When $p = -1$, by Proposition 2.29,

$$E_{-1,q}^2(S) \cong H_0^{\text{FI}}\left(H_q(F(M))\right)_S.$$

Since $n \geq 2$ and M is connected, the configuration space $F_k(M)$ is connected for all $k \geq 1$, and there is an isomorphism of FI-modules $H_0(F(M)) \cong M(0)$. Therefore when $q = 0$,

$$E_{p,0}^2(S) \cong H_p\left(\text{Inj}_*(H_0(F(M)))\right)_S \cong H_p\left(\text{Inj}_*(M(0))\right)_S \cong \tilde{H}_p\left(\text{Inj}_*(S)\right),$$

a group that is nonzero only when $p = |S| - 1$ by Theorem 2.18. When $|S| = k$, the $\text{FI}_\#$ -module structure on $H_q(F(M))$ and Theorem 2.26 imply that the E^2 -page has the form

$$E_{p,q}^2(k) \cong H_p\left(\text{Inj}_*(H_q(F_k(M)))\right) \cong \text{Ind}_{\mathfrak{S}_{p+1} \times \mathfrak{S}_{k-p-1}}^{\mathfrak{S}_k} H_{p+1}(\text{Inj}_*(p+1)) \boxtimes H_0^{\text{FI}}(H_q(F(M)))_{k-p-1}$$

as claimed. \square

Theorem 3.11. *Let M be a noncompact connected manifold of dimension at least two. Then*

$$\deg H_0^{\text{FI}}(H_i(F(M))) \leq 2i.$$

When M is orientable, Theorem 3.11 is a result of Church, Ellenberg, and Farb [CEF15, Theorem of 6.4.3] proved by different methods. They use the Leray spectral sequence associated to the inclusion $F_k(M) \hookrightarrow M^k$, described by Totaro [Tot96].

Proof of Theorem 3.11. Consider the arc resolution spectral sequence described in Proposition 3.10. For $p + q \leq |S| - 2$, Proposition 3.8 implies that the sequence converges to zero:

$$E_{p,q}^\infty(S) \cong H_{p+q+1}(F_S(M), ||\text{Arc}_\bullet(F_S(M))||) \cong 0 \quad \text{for } p + q \leq |S| - 2.$$

We now prove Theorem 3.11 by induction on homological degree i . Observe that

$$\deg H_0^{\text{FI}}(H_0(F(M))) = 0$$

since $H_0(F(M)) \cong M(0)$. Assume that $\deg H_0^{\text{FI}}(H_q(F(M))) \leq 2q$ for all $q < i$. Using Theorem 2.27 and our inductive hypothesis, we obtain

$$E_{p,q}^2(S) = 0 \quad \text{for } p \leq |S| - 2 - 2q \quad \text{and} \quad q < i$$

(equivalently $|S| \geq p + 2(q + 1)$). This shows that there are no possible differentials into (or out of) $E_{-1,i}^r(S)$ for $r > 1$ and $|S| > 2i$. See Figure 18.

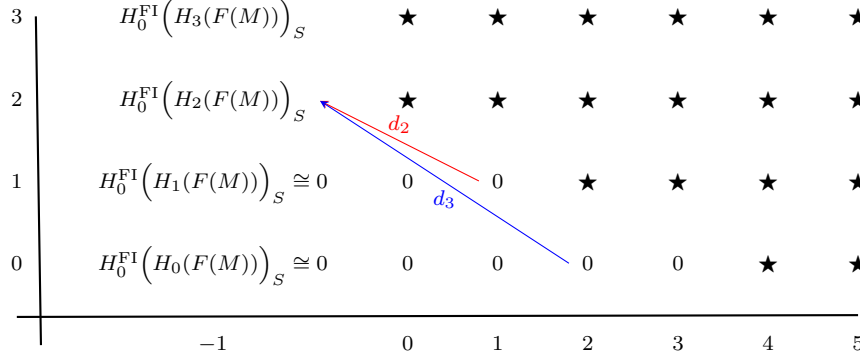


Figure 18: $E_{p,q}^2(S)$ in the inductive step, illustrated for $|S| = 5$ and $i = 2$.

Thus for $|S| > 2i$,

$$H_0^{\text{FI}}(H_i(F(M)))_S \cong E_{-1,i}^2(S) \cong E_{-1,i}^\infty(S) \cong 0.$$

This shows that $\deg H_0^{\text{FI}}(H_i(F(M))) \leq 2i$. The claim now follows by induction. \square

Remark 3.12. If M or N does not have a smooth structure, then the space of smooth embeddings of N into M is not well-defined. To construct the stabilization maps and homology operations of Section 3.1, it suffices to consider the space of all continuous embeddings with the compact open topology.

In the appendix to Kupers–Miller [KM14], the manifolds are assumed to be smooth and the space of embeddings is topologized with the C^∞ topology. To modify the arguments of [KM14] to prove a version of Theorem 3.7 which applies to topological manifolds, we need to consider a space of embeddings that satisfies a parameterized isotopy extension theorem (see Burghelea–Lashof [BL74, Page 19]). The space of all continuous embeddings topologized with the compact open topology is not the correct space to consider. One space of embeddings of topological manifolds that is compatible with the proof in [KM14] is the following: Let $\text{Emb}_\bullet^{lf}(N, M)$ denote the simplicial set whose space of k -simplices is the set of locally flat embeddings of $\Delta^k \times N$ into $\Delta^k \times M$ that commute with the projection to Δ^k . Using $||\text{Emb}_\bullet^{lf}(N, M)||$ in the definition of the arc resolution allows us to apply the arguments of [KM14] to topological manifolds without significant modifications.

3.3 Differentials in the arc resolution spectral sequence

The goal of this subsection is to compute many of the differentials in the arc resolution spectral sequence. This calculation will be used in the subsequent two subsections to prove secondary representation stability in dimension 2 and an improved representation stability range in higher dimensions.

Definition 3.13. Let M be the interior of an n -manifold \overline{M} with an embedding $[0, 1] \hookrightarrow \partial \overline{M}$. Choose an interval in the boundary of the half-closed disk $\mathbb{R}^{n-1} \times (-\infty, 0]$. Fix an embedding

$$\bar{e} : \overline{M} \sqcup (\mathbb{R}^{n-1} \times (-\infty, 0]) \hookrightarrow \overline{M}$$

such that:

- On the interior of the domain, \bar{e} restricts to an embedding of $M \sqcup \mathbb{R}^n$ into M such that $\bar{e}|_M$ is isotopic to the identity.
- The embedding \bar{e} restricts to an embedding of the two boundary intervals of \bar{M} and $\mathbb{R}^{n-1} \times (-\infty, 0]$ into the boundary interval of \bar{M} .

Then the embedding \bar{e} induces a map of spaces

$$e_{p,p'} : \text{Arc}_p(F_S(\mathbb{R}^n)) \times \text{Arc}_{p'}(F_T(M)) \rightarrow \text{Arc}_{p+p'+1}(F_{S \sqcup T}(M))$$

as in Figure 19.

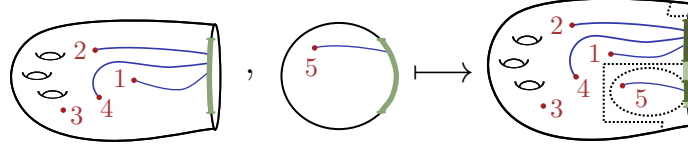


Figure 19: The map $e_{2,0} : \text{Arc}_2(F_{\{1,2,3,4\}}(\mathbb{R}^n)) \times \text{Arc}_0(F_{\{5\}}(M)) \rightarrow \text{Arc}_3(F_{\{1,2,3,4,5\}}(M))$.

In this subsection, the symbol $C_i(X)$ will denote the i dimensional singular chains on a space X , as opposed to the configuration space of i unordered particles. Let $\partial : C_i(X) \rightarrow C_{i-1}(X)$ denote the usual boundary operator. Due to the abundance of the letter “ d ” in this subsection, we will denote maps on singular chains induced by face maps in the arc resolution by f_i . The differential d^1 of the arc resolution spectral sequence is given by the alternating sum of the maps in homology induced by the face maps. We will denote the map on singular chains given by the alternating sum of the face maps by d^1 as well.

Remark 3.14. Given an augmented semi-simplicial space A_\bullet , the geometric realization spectral sequence is defined to be the relative filtered space spectral sequence associated to the skeletal filtration of $\|A_\bullet\|$. There are, however, other ways to construct spectral sequences from A_\bullet . The groups $C_p(A_q)$ form a double complex, with a (vertical) differential given by the boundary map for singular chains, and a (horizontal) differential induced by the alternating sum of the face maps. Both of the spectral sequences associated to this double complex converge to the relative homology groups $H_{p+q+1}(A_{-1}, \|A_\bullet\|)$. Moreover, one of them agrees with the geometric realization spectral sequence starting on the E^1 -page; see for example Bendersky and Gitler [BG91, Proposition 1.2].

In particular, we may re-define the E^0 -page of the arc resolution spectral sequence to be the complex $E_{p,q}^0(k) \cong C_q(\text{Arc}_p(F_k(M)))$ shown in Figure 20. This formulation of the spectral sequence will be

3	$C_3(\text{Arc}_{-1}(F_k(M)))$	$\xleftarrow[f_0]{d^1}$	$C_3(\text{Arc}_0(F_k(M)))$	$\xleftarrow[f_0 - f_1]{d^1}$	$C_3(\text{Arc}_1(F_k(M)))$	$\xleftarrow[f_0 - f_1 + f_2]{d^1}$	$C_3(\text{Arc}_2(F_k(M)))$	$\xleftarrow{\quad}$
	$\downarrow \partial$		$\downarrow \partial$		$\downarrow \partial$		$\downarrow \partial$	
2	$C_2(\text{Arc}_{-1}(F_k(M)))$	$\xleftarrow[f_0]{d^1}$	$C_2(\text{Arc}_0(F_k(M)))$	$\xleftarrow[f_0 - f_1]{d^1}$	$C_2(\text{Arc}_1(F_k(M)))$	$\xleftarrow[f_0 - f_1 + f_2]{d^1}$	$C_2(\text{Arc}_2(F_k(M)))$	$\xleftarrow{\quad}$
	$\downarrow \partial$		$\downarrow \partial$		$\downarrow \partial$		$\downarrow \partial$	
1	$C_1(\text{Arc}_{-1}(F_k(M)))$	$\xleftarrow[f_0]{d^1}$	$C_1(\text{Arc}_0(F_k(M)))$	$\xleftarrow[f_0 - f_1]{d^1}$	$C_1(\text{Arc}_1(F_k(M)))$	$\xleftarrow[f_0 - f_1 + f_2]{d^1}$	$C_1(\text{Arc}_2(F_k(M)))$	$\xleftarrow{\quad}$
	$\downarrow \partial$		$\downarrow \partial$		$\downarrow \partial$		$\downarrow \partial$	
0	$C_0(\text{Arc}_{-1}(F_k(M)))$	$\xleftarrow[f_0]{d^1}$	$C_0(\text{Arc}_0(F_k(M)))$	$\xleftarrow[f_0 - f_1]{d^1}$	$C_0(\text{Arc}_1(F_k(M)))$	$\xleftarrow[f_0 - f_1 + f_2]{d^1}$	$C_0(\text{Arc}_2(F_k(M)))$	$\xleftarrow{\quad}$
	$\downarrow \partial$		$\downarrow \partial$		$\downarrow \partial$		$\downarrow \partial$	
	0		1		2		3	

Figure 20: $E_{p,q}^0(k) \cong C_q(\text{Arc}_p(F_k(M)))$.

convenient for the calculations we perform in this section. See Bott and Tu [BT82, Formula 14.12, Page 164] for a construction of the differentials in the spectral sequence of a double complex.

Convention 3.15. We can represent chains in $E_{p,q}^0(k) \cong C_q(\text{Arc}_p(F_k(M)))$ by q -parameter families of points in $\text{Arc}_p(F_k(M))$, for example, Figure 21 shows a map $[0, 1]^2 \rightarrow \text{Arc}_0(F_5M)$. Given any subdivision of the product $[0, 1]^2$ into triangles, we can express this map as a linear combination of singular chains. We interpret Figure 21 to represent the associated chain in $C_2(\text{Arc}_0(F_5M))$ or its homology class in $H_2(\text{Arc}_0(F_5M))$. To compute the boundary of a chain $s : [0, 1]^q \rightarrow \text{Arc}_p(F_k(M))$, we

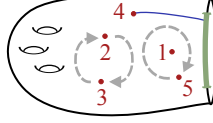


Figure 21: A map $[0, 1]^2 \rightarrow \text{Arc}_0(F_5M)$. As the first factor $[0, 1]$ ranges from 0 to 1, particle 3 moves from bottom to top while simultaneously particle 2 moves from top to bottom. As the second factor $[0, 1]$ ranges from 0 to 1, particle 5 moves in a closed loop around particle 1.

choose an order on the q factors $e_i : [0, 1] \rightarrow \text{Arc}_p(F_k(M))$ of the domain and use the formula

$$\partial s = \sum_j (-1)^{j+1} e_1 \times e_2 \times \cdots \times \partial(e_j) \times \cdots \times e_q \quad \text{where} \quad \partial e_i = e_i(1) - e_i(0)$$

For example, if we order the two singular 1-simplices in Figure 21 as they appear left to right, and observe that the second singular 1-simplex is a cycle, we find that the boundary is the chain shown in Figure 22. More generally, given the product of a singular 1-simplex $e : [0, 1] \rightarrow \text{Arc}_p(F_k(M))$ and any

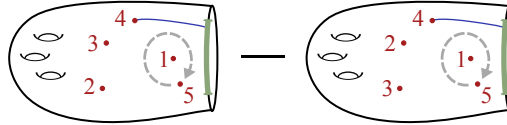


Figure 22: The boundary of Figure 21.

chain $y \in C_{q-1}(\text{Arc}_p(F_k(M)))$, we order their product $(e \times y)$, and so their boundary is given by the $(q-1)$ -chain

$$\partial(e \times y) = ((\partial e \times y) - (e \times \partial y)) = ((e(1) \times y) - (e(0) \times y) - (e \times \partial y)).$$

For example, the product chain shown in Figure 23a has boundary given by the chain depicted in in Figure 23b. These conventions will feature in the computations carried out below and are important

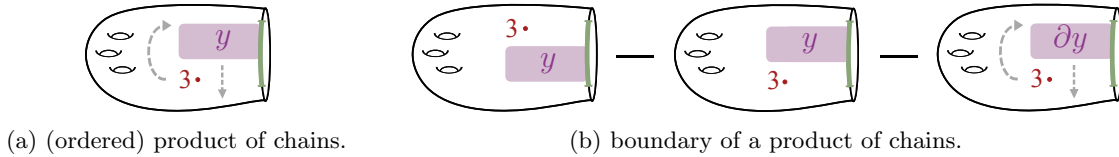


Figure 23: The boundary of a product of chains.

for determining signs.

The main result of this section is the values of the differentials computed in the following lemma. This result, combined with the Leibniz rule stated in Lemma 3.18, determines a large portion of the differentials in the spectral sequence associated to the arc resolution of M .

Lemma 3.16. *Let $E_{p,q}^r(S)$ be the arc resolution spectral sequence and let $k = |S|$. Consider an element of Reutenauer's basis for \mathcal{L}_S (Theorem 2.36).*

$$L = [[[\cdots [a_1, a_2], a_3], \dots], a_{k-1}], a_k] \in E_{k-1,0}^1(S).$$

Then $d^r(L) = 0$ for $r < k$ and

$$d^k(L) = t_{\psi(\dots\psi(\psi(\psi(a_1, a_2), a_3), a_4), \dots, a_k)}(y_0).$$

Here y_0 denotes the class of a point in $H_0(F_0(M))$.

For example, the image of the element $[[[1, 2], 3], 4] \in E_{3,0}^1(4)$ is shown in Figure 24. The particles labeled 2, 3, and 4 orbit counterclockwise around the particle labeled 1 in concentric circles.



Figure 24: $d^4([[[1, 2], 3], 4])$.

Proof of Lemma 3.16. We proceed by induction on k . We begin with the base case $k = 1$, $S = \{1\}$, where we observe that d^1 differential maps the singleton word $1 \in E_{0,0}^1$ to the class of the point in $H_0(\text{Arc}_{-1}(F_1(M))) = H_0(F_1(M))$. This result, shown in Figure 25, follows from the description of the spectral sequence in Proposition 3.10.

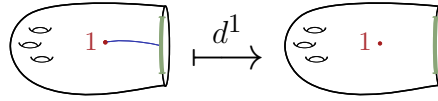


Figure 25: $d^1(1)$.

Now suppose that $k > 1$, and let x_{k-1} be the class $[[\dots[a_1, a_2], a_3], \dots], a_{k-1}] \in E_{k-2,0}^1(S \setminus \{a_k\})$. Suppose by induction that x_{k-1} survives to $E_{k-1,0}^{k-1}(S)$, and

$$d^{k-1}(x_{k-1}) = t_{\psi(\dots\psi(\psi(\psi(a_1, a_2), a_3), a_4), \dots, a_{k-1})}(y_0).$$

This means that there exist chains x_{k-2}, \dots, x_1 with $x_i \in E_{i-1, k-i-1}^0$, such that $d^1(x_i) = (-1)^{i-1} \partial(x_{i-1})$, as in Figure 26 (compare to Bott–Tu [BT82, Formula 14.12, Page 164]).

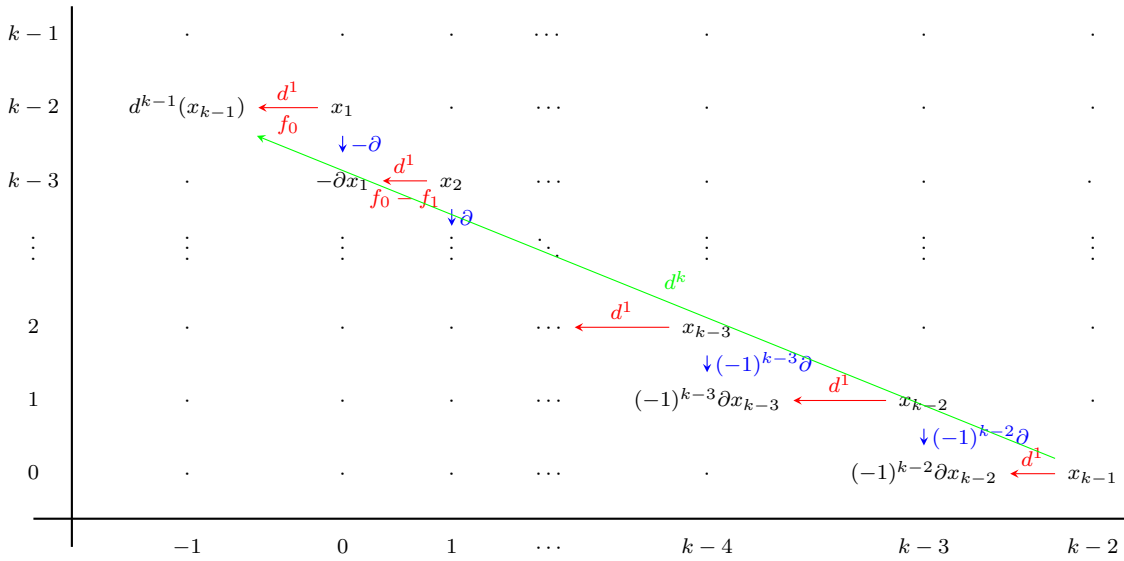


Figure 26: Computing $d^{k-1}(x_{k-1}) = t_{\psi(\dots\psi(\psi(\psi(a_1, a_2), a_3), a_4), \dots, a_{k-1})}(y_0)$.

The class x_i is a chain on a space with i arcs. We show, by induction, that the classes x_i can be taken to be in the image of the map induced by the map $e_{k-i-1,-1}$ of Definition 3.17, so they may be represented as in Figure 27.

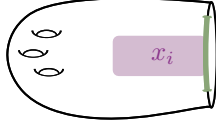


Figure 27: The particles of the chain x_i can be taken to be in the shaded box.

We may assume without loss of generality that the label a_k is the letter k . Now consider the class

$$L = [x_{k-1}, k] \in E_{k-1,0}^1(S)$$

as shown in Figure 28.

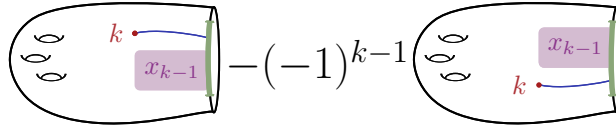


Figure 28: A chain representing $[x_{k-1}, k]$.

Our goal is to show that $d^r([x_{k-1}, k]) = 0$ for $r < k$, and to compute $d^k([x_{k-1}, k])$. To do this, we will compute a zigzag of chains $\xi_i \in E_{i,k-i-1}^0$ satisfying $(-1)^i \partial(\xi_{i-1}) = d^1(\xi_i)$, beginning with $\xi_{k-1} = L$. The image

$$d^1([x_{k-1}, k]) = \sum (-1)^i f_i([x_{k-1}, k])$$

is shown in Figure 29.

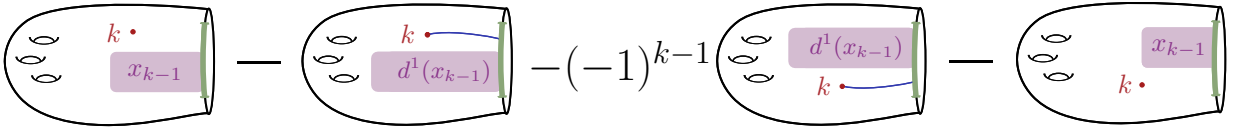


Figure 29: A chain representing $\sum (-1)^i f_i([x_{k-1}, k])$.

Then $\sum (-1)^i f_i([x_{k-1}, k])$ is equal to the boundary $(-1)^{k-1} \partial(\xi_{k-2})$, where ξ_{k-2} is the chain shown in Figure 30. Recall that x_{k-2} is defined such that $(-1)^{k-2} \partial(x_{k-2}) = d^1(x_{k-1}) = \sum (-1)^i f_i(x_{k-1})$, and that $\partial(x_{k-1}) = 0$. In Figure 30, and in the images throughout this proof, we will order the simplices with the simplex designated by the dotted line first, and the class x_i in the shaded region second, so the boundary is computed as in Figure 23. We have shown that $d^1([x_{k-1}, k])$ is zero in homology, and $[x_{k-1}, k]$ survives to E^2 .

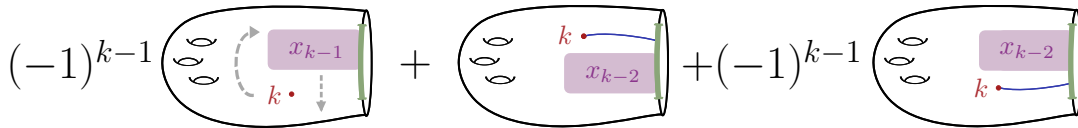


Figure 30: The chain ξ_{k-2} .

We now proceed by induction on the degree r of the page E^r . Suppose by induction that $d^{r-1}([x_{k-1}, k])$ is represented by the boundary $(-1)^{k-r-1} \partial(\xi_{k-r})$, where the chain ξ_{k-r} is shown in Figure 31. Then $d^r([x_{k-1}, k]) = \sum (-1)^i f_i(\xi_{k-r})$ is shown in Figure 32. If $r \leq k-2$, then by inductive hypothesis there is a chain x_{k-r-1} with

$$(-1)^{k-r-1} \partial x_{k-r-1} = \sum (-1)^i f_i(x_{k-r}).$$

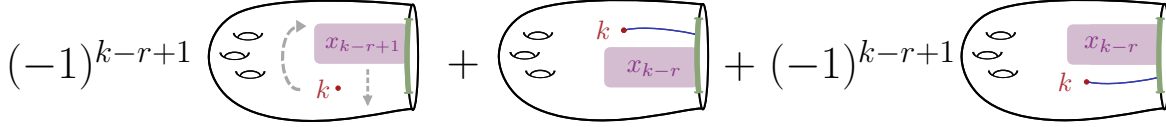


Figure 31: The chain ξ_{k-r} .

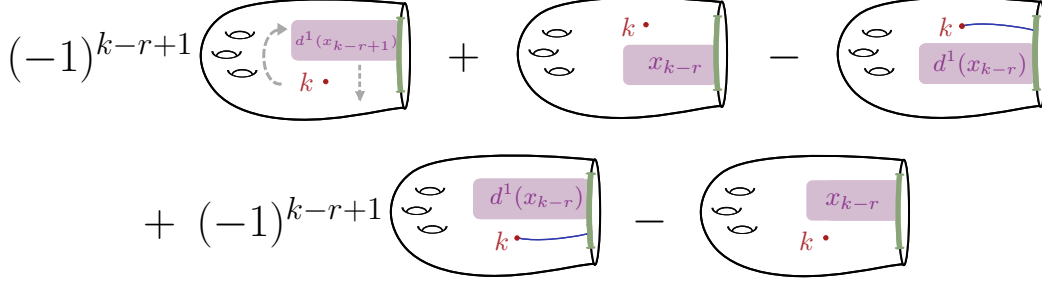


Figure 32: The chain $d^r([x_{k-1}, k]) = \sum (-1)^i f_i(\xi_{k-r})$ for $r \leq k-1$.

In this case, the chain ξ_{k-r-1} in Figure 33 is such that $(-1)^{k-r-1} \partial(\xi_{k-r-1})$ equals the chain representing $d^r([x_{k-1}, k])$ in Figure 32, and so $d^r([x_{k-1}, k]) = 0$ on E^r . By comparing Figure 33 to Figure 31, we see we have completed the inductive step in the induction on r .

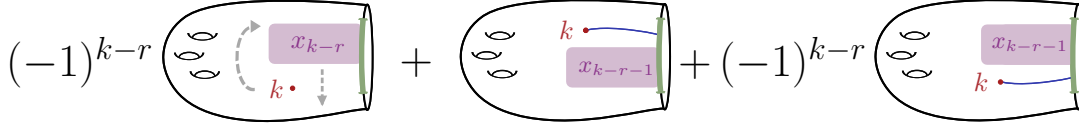


Figure 33: A class ξ_{k-r-1} with $(-1)^{k-r-1} \partial(\xi_{k-r-1}) = \sum (-1)^i f_i(\xi_{k-r})$ for $r \leq k-2$.

Now consider Figure 32 when $r = k-1$. There are no arcs attached in the class

$$d^1(x_1) = \sum (-1)^i f_i(x_1) = f_0(x_1),$$

and by induction $d^1(x_1) = d^{k-1}(x_{k-1})$ is a ∂ -cycle. Hence the class $\sum (-1)^i f_i([x_{k-1}, k])$ is the boundary of the class in Figure 34. Again, we conclude that $d^{k-1}([x_{k-1}, k]) = 0$. We can compute $d^k([x_{k-1}, k])$

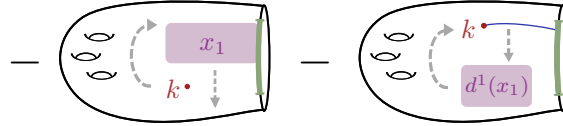


Figure 34: A chain ξ_0 with boundary $-d^{k-1}([x_{k-1}, k]) = -\sum (-1)^i f_i(\xi_1)$.

by applying the map induced by the alternating sum of face maps to Figure 34, with the result shown in Figure 35. By construction:

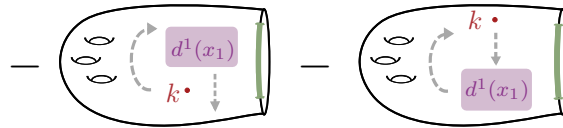


Figure 35: The image $d^k([x_{k-1}, k])$.

$$d^1(x_1) = d^{k-1}(x_{k-1}) = t_{\psi(\dots\psi(\psi(\psi(a_1, a_2), a_3), a_4), \dots, a_{k-1})}(y_0).$$

Hence the chain in Figure 35 is homologous to the chain in Figure 36. In this figure we have negated the chain by reversing the direction of the arrow from clockwise to counterclockwise. Figure 36 concludes

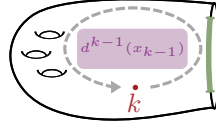


Figure 36: The image $d^k([x_{k-1}, k])$.

the induction on k , and the proof. \square

Although the statement of Lemma 3.16 only involves differentials out of the bottom row of the arc resolution spectral sequence, the lemma actually allows us to compute a very large portion of the differentials because the differentials in the arc resolution spectral sequence satisfies a Leibniz rule.

Definition 3.17. Suppose N is the interior of a manifold \bar{N} with nonempty boundary, with a choice of embedded interval in $\partial\bar{N}$. We let $E_{p,q}^r[N](S)$ denote the associated term in then arc resolution spectral.

Lemma 3.18. The maps $e_{p,p'}$ of Definition 3.17 induce maps

$$t^r : E_{p,q}^r[\mathbb{R}^n](S) \otimes E_{p',q'}^r[\mathbb{R}^n](T) \rightarrow E_{p+p'-1, q+q'}^r[M](S \sqcup T)$$

These maps satisfy the following Leibniz rule with respect to the differentials; if $a \in E_{p,q}^r[\mathbb{R}^n](S)$ and $b \in E_{p',q'}^r[M](S)$, then:

$$d^r(t^r(a, b)) = t^r(d^r(a), b) + (-1)^{p+q} t^r(a, d^r(b))$$

Proof. The geometric realization spectral sequence can be realized as the spectral sequence of a filtered chain complex. The maps $e_{p,p'}$ induce maps of filtered chain complexes. Maps of filtered chain complexes induce maps of filtered chain complex spectral sequences which satisfy the Leibniz rule with respect to the differentials (see for example notes of Galatius [Gal, Theorem 9.5]). \square

3.4 Proof of secondary representation stability

In this subsection, we prove Theorem 1.4, secondary representation stability for the homology of configuration spaces. For this result we need to assume R is a field of characteristic zero. The reason for this assumption is so that we can apply corollaries of Theorem 2.14. Assuming that R is a field also makes the formulation of Proposition 3.24 easier, although workarounds do exist for general rings. We will also assume that the manifold M has finite type. This implies that the homology groups of the ordered configuration spaces are finitely generated as abelian groups. The $\bigwedge(\text{Sym}^2 R)$ -modules which we will show exhibit secondary representation stability are defined as follows.

Definition 3.19. Let $\mathcal{W}_i^M(S) = H_0^{\text{FI}}\left(H_{\frac{|S|+i}{2}}(FM)\right)_S$. Here we use the convention that fractional homology groups are 0.

The collection of \mathfrak{S}_k -representations $\mathcal{W}_k^M(k)$ assemble to form a $\bigwedge(\text{Sym}^2 R)$ -module as follows. Let $(f, Q) \in \text{Hom}_{\text{FIM}^+}(S, T)$ be a basis vector with $f : S \rightarrow T$ an injective map and $Q = \{(x_1, y_1), \dots, (x_d, y_d)\}$ an oriented matching of the complement of the image. Let $f' : S \rightarrow f(S)$ be the bijective map defined by f and let

$$f'_* : H_{i+\frac{|S|}{2}}(F(M))_S \rightarrow H_{i+\frac{|S|}{2}}(F(M))_{f(S)}$$

be the map on homology induced by the FI-modules structure. The element (f, Q) acts on a homology class in $H_0^{\text{FI}}(H_{i+|S|/2}(F(M)))_S$ by $t_{\psi(x_d, y_d)} \circ \dots \circ t_{\psi(x_1, y_1)} \circ f'_*$, as shown in Figure 7. Although this map is defined on the homology of configuration spaces, it descends to a map on its generators $H_0^{\text{FI}}(H_*(F(M)))$. The order of the composition factors $t_{\psi(x_i, y_i)}$ only affects the sign of the homology class: this sign is exactly what differentiates the category FIM^+ from the linearization of FIM.

To prove secondary representation stability we will need to better understand the algebraic structure on the E^2 -page of the arc resolution spectral sequence. To that end, we now define a filtration on the top homology of the complex of injective words.

Definition 3.20. For d and b of the same parity, let \mathcal{T}_d^b be the image of the natural map:

$$\text{Ind}_{\mathfrak{S}_b \times \mathfrak{S}_2 \times \dots \times \mathfrak{S}_2}^{\mathfrak{S}_d} \mathcal{T}_b \boxtimes \mathcal{L}_2 \boxtimes \dots \boxtimes \mathcal{L}_2 \rightarrow \mathcal{T}_d$$

By Proposition 3.21 below, we can identify the groups \mathcal{T}_d^b with the groups:

$$\text{Ind}_{\mathfrak{S}_b \times \mathfrak{S}_2 \times \dots \times \mathfrak{S}_2}^{\mathfrak{S}_d} \mathcal{T}_b \boxtimes \mathcal{L}_2 \boxtimes \dots \boxtimes \mathcal{L}_2.$$

For example, \mathcal{T}_7^3 is the span of the elements:

$$\left\{ \begin{array}{l} [[a, b], c] [d, e] [f, g], \quad [[a, c], b] [d, e] [f, g] \end{array} \mid \text{decompositions } [7] = \{a, b, c\} \sqcup \{d, e\} \sqcup \{f, g\} \right\}.$$

Proposition 3.21. The map $\text{Ind}_{\mathfrak{S}_b \times \mathfrak{S}_2 \times \dots \times \mathfrak{S}_2}^{\mathfrak{S}_d} \mathcal{T}_b \boxtimes \mathcal{L}_2 \boxtimes \dots \boxtimes \mathcal{L}_2 \rightarrow \mathcal{T}_d$ defining the group \mathcal{T}_d^b is injective.

Proof. We must show that the module

$$\text{Ind}_{\mathfrak{S}_b \times \mathfrak{S}_2 \times \dots \times \mathfrak{S}_2}^{\mathfrak{S}_d} \mathcal{T}_b \boxtimes \mathcal{L}_2 \boxtimes \dots \boxtimes \mathcal{L}_2 \cong \bigoplus_{\substack{[d] = B \sqcup A_1 \sqcup A_2 \sqcup \dots \sqcup A_{\frac{d-b}{2}} \\ |B|=b, |A_i|=2}} \mathcal{T}_B \otimes \mathcal{L}_{A_1} \otimes \dots \otimes \mathcal{L}_{A_{\frac{d-b}{2}}}$$

injects into \mathcal{T}_d . The summand indexed by the set decomposition $[d] = B \sqcup A_1 \sqcup A_2 \sqcup \dots \sqcup A_{\frac{d-b}{2}}$ embeds as the span of injective words:

$$\left\{ L[a_1, b_1][a_2, b_2] \cdots \left[a_{\frac{d-b}{2}}, b_{\frac{d-b}{2}} \right] \mid A_i = \{a_i, b_i\}, \quad L \in \mathcal{T}_B \right\}.$$

Given any element in the image of this summand – viewed as a linear combination of injective words – and given any word w appearing as a term in this element, we can uniquely recover the decomposition $[d] = B \sqcup A_1 \sqcup A_2 \sqcup \dots \sqcup A_{\frac{d-b}{2}}$ by observing the order of the letters $[d]$ in w . Hence the intersection of the image of distinct summands is zero, and the map is injective as claimed. \square

Proposition 3.22. There is a short exact sequence:

$$0 \longrightarrow \mathcal{T}_d^b \longrightarrow \mathcal{T}_d^{b+2} \longrightarrow \text{Ind}_{\mathfrak{S}_{b+2} \times \mathfrak{S}_2 \times \dots \times \mathfrak{S}_2}^{\mathfrak{S}_d} (\mathcal{T}_{b+2}/\mathcal{T}_{b+2}^b) \boxtimes \mathcal{L}_2 \boxtimes \dots \boxtimes \mathcal{L}_2 \longrightarrow 0.$$

Proof. By Proposition 3.21, we may identify:

$$\begin{aligned} \mathcal{T}_d^b &\cong \text{Ind}_{\mathfrak{S}_b \times \mathfrak{S}_2 \times \dots \times \mathfrak{S}_2}^{\mathfrak{S}_d} \mathcal{T}_b \boxtimes \mathcal{L}_2 \boxtimes \dots \boxtimes \mathcal{L}_2 \\ &\cong \text{Ind}_{\mathfrak{S}_{b+2} \times \mathfrak{S}_2 \times \dots \times \mathfrak{S}_2}^{\mathfrak{S}_d} \left(\text{Ind}_{\mathfrak{S}_b \times \mathfrak{S}_2}^{\mathfrak{S}_{b+2}} \mathcal{T}_b \boxtimes \mathcal{L}_2 \right) \boxtimes \mathcal{L}_2 \boxtimes \dots \boxtimes \mathcal{L}_2 \\ &\cong \text{Ind}_{\mathfrak{S}_{b+2} \times \mathfrak{S}_2 \times \dots \times \mathfrak{S}_2}^{\mathfrak{S}_d} (\mathcal{T}_{b+2}^b) \boxtimes \mathcal{L}_2 \boxtimes \dots \boxtimes \mathcal{L}_2. \end{aligned}$$

$$\text{Moreover, } \mathcal{T}_d^{b+2} \cong \text{Ind}_{\mathfrak{S}_{b+2} \times \mathfrak{S}_2 \times \dots \times \mathfrak{S}_2}^{\mathfrak{S}_d} \mathcal{T}_{b+2} \boxtimes \mathcal{L}_2 \boxtimes \dots \boxtimes \mathcal{L}_2.$$

Since tensor product and induction are right-exact operations, from the short exact sequence

$$0 \longrightarrow \mathcal{T}_{b+2}^b \longrightarrow \mathcal{T}_{b+2} \longrightarrow (\mathcal{T}_{b+2}/\mathcal{T}_{b+2}^b) \longrightarrow 0 \quad (\text{exact by Proposition 3.21})$$

we can conclude that the sequence in question is exact at each point except possibly \mathcal{T}_d^b . But Proposition 3.21 implies that the composition of the maps

$$\mathcal{T}_d^b \longrightarrow \mathcal{T}_d^{b+2} \longrightarrow \mathcal{T}_d$$

is injective. This implies that the map $\mathcal{T}_d^b \longrightarrow \mathcal{T}_d^{b+2}$ injects, and we conclude that the sequence

$$0 \longrightarrow \mathcal{T}_d^b \longrightarrow \mathcal{T}_d^{b+2} \longrightarrow \text{Ind}_{\mathfrak{S}_{b+2} \times \mathfrak{S}_2 \times \dots \times \mathfrak{S}_2}^{\mathfrak{S}_d} (\mathcal{T}_{b+2}/\mathcal{T}_{b+2}^b) \boxtimes \mathcal{L}_2 \boxtimes \dots \boxtimes \mathcal{L}_2 \longrightarrow 0$$

is exact. \square

Definition 3.23. Let $E_{p,q}^r(k)$ denote entry (p, q) on the r th page of the arc resolution spectral sequence for the set $[k]$. For $i \geq 0$, let

$$A_j^i(k) = E_{2j-1, i-j+\lceil k/2 \rceil}^2(k).$$

The groups $A_*^i(k)$ with the d^2 differential form a chain complex which we call the “ i th even diagonal.” For $i \geq 0$, let

$$B_j^i(k) = E_{2j, i-j+\lceil k/2 \rceil}^2(k).$$

Call the chain complex $B_*^i(k)$ the “ i th odd diagonal.”

Some examples of these complexes on $E_{p,q}^2(6)$ and are illustrated in Figure 37. $A_*^3(k)$ will always be the third (counting from $i = 0$) diagonal above the triangle of zeroes, and similarly $B_*^1(k)$ is the first (counting from $i = 0$) offset diagonal above the triangle of zeroes.

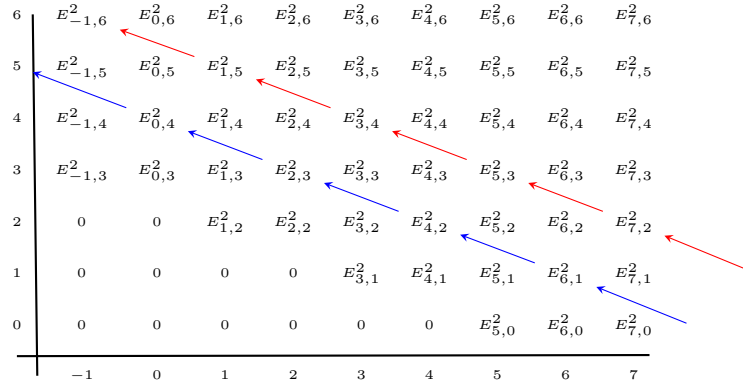


Figure 37: The complexes $A_*^3(6)$ (differentials in red) and $B_*^1(6)$ (in blue) on $E_{p,q}^2(6)$.

We now relate the chain complexes $A_*^i(k)$ and $B_*^i(k)$ to the chain complexes $\text{Inj}_*^2(\mathcal{W}_j^M)$. Note that $A_*^i(k)$ and $B_*^i(k)$ are 0 for $* < 0$ in contrast to $\text{Inj}_*^2(\mathcal{W}_j^M)$, which is potentially nonzero for $* = -1$. To simplify indexing in the following proposition, we introduce two $\bigwedge(\text{Sym}^2 R)$ -modules, $\mathcal{V}_i(k)$ and $\mathcal{U}_i(k)$. The collections of $\bigwedge(\text{Sym}^2 R)$ -modules $\{\mathcal{V}_i(k)\}_{i=0}^\infty$ and $\{\mathcal{U}_i(k)\}_{i=0}^\infty$ contain the same information as each other, and as $\{\mathcal{W}_i^M\}_{i=0}^\infty$.

Proposition 3.24. Suppose R is a field. Define $\bigwedge(\text{Sym}^2 R)$ -modules

$$\mathcal{V}_i(k) := H_0^{\text{FI}}(H_{i+\lceil \frac{k}{2} \rceil}(F(M)))_k \quad \text{and} \quad \mathcal{U}_i(k) := H_0^{\text{FI}}(H_{i+\lfloor \frac{k}{2} \rfloor}(F(M)))_k.$$

The chain complex $A_*^i(k)$ has a filtration by chain complexes such that the filtration quotients are isomorphic to

$$\text{Ind}_{\mathfrak{S}_{2b} \times \mathfrak{S}_{k-2b}}^{\mathfrak{S}_k} (\mathcal{T}_{2b}/\mathcal{T}_{2b}^{2b-2}) \boxtimes \left(\text{Inj}_{*-b-1}^2 \mathcal{V}_i \right)_{k-2b}.$$

The chain complex $B_*^i(k)$ has a filtration such that the filtration differences are isomorphic to

$$\text{Ind}_{\mathfrak{S}_{2b+1} \times \mathfrak{S}_{k-2b-1}}^{\mathfrak{S}_k} (\mathcal{T}_{2b+1}/\mathcal{T}_{2b+1}^{2b-1}) \boxtimes \left(\text{Inj}_{*-b}^2 \mathcal{U}_i \right)_{k-2b-1}.$$

Proof. By Proposition 3.22, there is a filtration of \mathcal{T}_d given by

$$\begin{aligned} 0 &\hookrightarrow \mathcal{T}_d^0 \hookrightarrow \dots \hookrightarrow \mathcal{T}_d^{d-4} \hookrightarrow \mathcal{T}_d^{d-2} \hookrightarrow \mathcal{T}_d^d = \mathcal{T}_d && \text{when } d \text{ is even, and} \\ 0 &= \mathcal{T}_d^1 \hookrightarrow \mathcal{T}_d^3 \hookrightarrow \dots \hookrightarrow \mathcal{T}_d^{d-4} \hookrightarrow \mathcal{T}_d^{d-2} \hookrightarrow \mathcal{T}_d^d = \mathcal{T}_d && \text{when } d \text{ is odd,} \end{aligned}$$

whose quotients are the groups:

$$\frac{\mathcal{T}_d^b}{\mathcal{T}_d^{b-2}} \cong \text{Ind}_{\mathfrak{S}_b \times \mathfrak{S}_2 \times \dots \times \mathfrak{S}_2}^{\mathfrak{S}_d} (\mathcal{T}_b/\mathcal{T}_b^{b-2}) \boxtimes \mathcal{L}_2 \boxtimes \dots \boxtimes \mathcal{L}_2.$$

Since R is a field, it follows that

$$E_{p,q}^2(S) \cong \bigoplus_{\substack{S=P \sqcup R, \\ |P|=p+1}} \mathcal{T}_{p+1}(P) \otimes H_0^{\text{FI}}(H_q(F(M)))_{k-p-1}$$

is filtered by the modules

$$\bigoplus_{\substack{S=P \sqcup R, \\ |P|=p+1}} \mathcal{T}_{p+1}^b(P) \otimes H_0^{\text{FI}}(H_q(F(M)))_{k-p-1} \quad \text{for } b \equiv p+1 \pmod{2}$$

with filtration quotients

$$\begin{aligned} & \bigoplus_{\substack{S=P \sqcup R, \\ |P|=p+1}} \left(\text{Ind}_{\mathfrak{S}_b \times \mathfrak{S}_2 \times \dots \times \mathfrak{S}_2}^{\mathfrak{S}_{p+1}} (\mathcal{T}_b / \mathcal{T}_b^{b-2}) \boxtimes \mathcal{L}_2 \boxtimes \dots \boxtimes \mathcal{L}_2 \right) \otimes H_0^{\text{FI}}(H_q(F(M)))_{k-p-1} \\ &= \text{Ind}_{\mathfrak{S}_b \times \mathfrak{S}_2 \times \dots \times \mathfrak{S}_2 \times \mathfrak{S}_{k-p-1}}^{\mathfrak{S}_k} (\mathcal{T}_b / \mathcal{T}_b^{b-2}) \boxtimes \mathcal{L}_2 \boxtimes \dots \boxtimes \mathcal{L}_2 \boxtimes H_0^{\text{FI}}(H_q(F(M)))_{k-p-1} \\ &= \text{Ind}_{\mathfrak{S}_b \times \mathfrak{S}_{k-b}}^{\mathfrak{S}_k} (\mathcal{T}_b / \mathcal{T}_b^{b-2}) \boxtimes \left(\text{Ind}_{\mathfrak{S}_2 \times \dots \times \mathfrak{S}_2 \times \mathfrak{S}_{k-p-1}}^{\mathfrak{S}_{k-b}} \mathcal{L}_2 \boxtimes \dots \boxtimes \mathcal{L}_2 \boxtimes H_0^{\text{FI}}(H_q(F(M)))_{k-p-1} \right). \end{aligned}$$

This means that the filtration differences for $A_j^i(k) = E_{2j-1, i-j+\lceil k/2 \rceil}^2(k)$ are given by

$$\begin{aligned} & \text{Ind}_{\mathfrak{S}_b \times \mathfrak{S}_{k-b}}^{\mathfrak{S}_k} (\mathcal{T}_b / \mathcal{T}_b^{b-2}) \boxtimes \left(\text{Ind}_{\mathfrak{S}_2 \times \dots \times \mathfrak{S}_2 \times \mathfrak{S}_{k-2j}}^{\mathfrak{S}_{k-b}} \mathcal{L}_2 \boxtimes \dots \boxtimes \mathcal{L}_2 \boxtimes H_0^{\text{FI}}(H_{i-j+\lceil k/2 \rceil}(F(M)))_{k-2j} \right) \quad (b \text{ even}) \\ &= \text{Ind}_{\mathfrak{S}_b \times \mathfrak{S}_{k-b}}^{\mathfrak{S}_k} (\mathcal{T}_b / \mathcal{T}_b^{b-2}) \boxtimes \left(\text{Inj}_{j-\frac{b}{2}-1}^2 \mathcal{V}_i \right)_{k-b}. \end{aligned}$$

Similarly, the filtration differences for $B_j^i(k) = E_{2j, i-j+\lceil k/2 \rceil}^2(k)$ are given by

$$\begin{aligned} & \text{Ind}_{\mathfrak{S}_b \times \mathfrak{S}_{k-b}}^{\mathfrak{S}_k} (\mathcal{T}_b / \mathcal{T}_b^{b-2}) \boxtimes \left(\text{Ind}_{\mathfrak{S}_2 \times \dots \times \mathfrak{S}_2 \times \mathfrak{S}_{k-2j-1}}^{\mathfrak{S}_{k-b}} \mathcal{L}_2 \boxtimes \dots \boxtimes \mathcal{L}_2 \boxtimes H_0^{\text{FI}}(H_{i-j+\lceil k/2 \rceil}(F(M)))_{k-2j-1} \right) \quad (b \text{ odd}) \\ &= \text{Ind}_{\mathfrak{S}_b \times \mathfrak{S}_{k-b}}^{\mathfrak{S}_k} (\mathcal{T}_b / \mathcal{T}_b^{b-2}) \boxtimes \left(\text{Inj}_{j-\frac{b}{2}-\frac{1}{2}}^2 \mathcal{U}_i \right)_{k-b}. \end{aligned}$$

For simplicity, we will reindex these filtrations by replacing odd values of b with $2b+1$ and even values of b with $2b$.

Let $\mathcal{F}_b(A_*^i)$ be the portion of the filtration of A_*^i constructed above containing element of the form $\text{Ind}_{\mathfrak{S}_{2b} \times \mathfrak{S}_{k-2b}}^{\mathfrak{S}_k} \mathcal{T}_{2b} \boxtimes (\text{Inj}_{j-b-1}^2 \mathcal{V}_i)_{k-2b}$ and similarly define $\mathcal{F}_b(B_*^i)$. We have constructed filtrations on the groups A_*^i and B_*^i , and now it remains to check that these are filtrations of chain complexes. We must verify that the boundary maps on the filtration quotients induced by the d^2 differential in the spectral sequence agree with the boundary maps of

$$\text{Ind}_{\mathfrak{S}_{2b} \times \mathfrak{S}_{k-2b}}^{\mathfrak{S}_k} (\mathcal{T}_{2b} / \mathcal{T}_{2b}^{2b-2}) \boxtimes (\text{Inj}_{*}^2 \mathcal{V}_i)_{k-2b} \quad \text{and of} \quad \text{Ind}_{\mathfrak{S}_{2b+1} \times \mathfrak{S}_{k-2b-1}}^{\mathfrak{S}_k} (\mathcal{T}_{2b+1} / \mathcal{T}_{2b+1}^{2b-1}) \boxtimes (\text{Inj}_{*}^2 \mathcal{U}_i)_{k-2b-1}.$$

First we will show that the subgroups $\mathcal{F}_b(A_*^i)$ are in fact subchain complexes. An element of $\mathcal{F}_b(A_j^i)(k)$ can be written as a sum of elements of the form:

$$t \boxtimes l_0 \boxtimes \dots \boxtimes l_{j-b} \boxtimes v \quad \text{with } t \in \mathcal{T}_{2b}, l_q \in \mathcal{L}_2 \text{ and } v \in H_0^{\text{FI}}(H_{i-j+\lceil \frac{k-1}{2} \rceil}(F(M)))_{2j-1}.$$

Moreover, we may assume that t is a product of Lie polynomials (and not a linear combination of products of Lie polynomials). By the Leibniz rule (Lemma 3.18) and our calculations of the differentials in the arc resolution spectral sequence from Lemma 3.16, it follows that the differential

$$d^2(t \boxtimes l_0 \boxtimes \dots \boxtimes l_{j-b} \boxtimes v)$$

is given by a signed sum of terms which remove one \mathcal{L}_2 factor and then stabilize v by the appropriate Browder operation. Since all of the terms in the sum are in $\mathcal{F}_b(A_{j-1}^i)(k)$, this establishes that that

$\mathcal{F}_b(A_*^i)$ is a filtration of chain complexes. There are two types of terms in the signed sum, the first involves removing an \mathcal{L}_2 factor from t and the second involve deleting one of the l_q 's. The portion of the sum involving terms of the second type is exactly the boundary map of the chain complex

$$\text{Ind}_{\mathfrak{S}_{2b} \times \mathfrak{S}_{k-2b}}^{\mathfrak{S}_k} \mathcal{T}_{2b} \boxtimes (\text{Inj}_*^2 \mathcal{V}_i)_{k-2b}.$$

Thus, it suffices to show that the portion of the sum involving terms of the first type are in $\mathcal{F}_{b-1}(A_{j-1}^i)(k)$, and hence zero in the quotient. This follows from the fact that if you remove an \mathcal{L}_2 factor from t , the resulting Lie polynomial will be two letters shorter and hence in \mathcal{T}_{2b-2} . A similar argument works for B_*^i . \square

The following result shows that vanishing on the E^3 -page of the arc resolution spectral sequence implies secondary representation stability.

Proposition 3.25. *There are isomorphisms of FB-modules:*

$$H_0(A_*^i) \cong H_0^{\text{FIM}^+}(\mathcal{V}_i).$$

There are isomorphisms of symmetric group representations:

$$H_0(A_*^i)(2k) \cong H_0^{\text{FIM}^+}(\mathcal{W}_{2i}^M)(2k) \quad \text{and} \quad H_0(A_*^i)(2k+1) \cong H_0^{\text{FIM}^+}(\mathcal{W}_{2i+1}^M)(2k+1).$$

Proof. Let \mathcal{Q} be a $\bigwedge(\text{Sym}^2 R)$ -module. In analogy to Proposition 2.29, there is an isomorphism

$$H_{-1}(\text{Inj}_*^2(\mathcal{Q})) \cong H_0^{\text{FIM}^+}(\mathcal{Q}).$$

By Proposition 3.24, the map $\text{Inj}_{*-1}^2(\mathcal{V}_i) \rightarrow A_*^i$ is an isomorphism for $* = 0, 1$. Thus,

$$H_{-1}(\text{Inj}_*^2(\mathcal{V}_i)) \cong H_0(A_*^i).$$

The second pair of isomorphisms follow from the fact that $\mathcal{V}_i \cong \mathcal{W}_{2i}^M \oplus \mathcal{W}_{2i+1}^M$. \square

We now prove the main theorem: if M is a finite type noncompact connected manifold, and R is a field of characteristic zero, then \mathcal{W}_i^M is a finitely generated $\bigwedge(\text{Sym}^2 R)$ -module.

Proof of Theorem 1.4. We will prove by induction that the $\bigwedge(\text{Sym}^2 R)$ -modules \mathcal{U}_i and \mathcal{V}_i defined in Proposition 3.24 are finitely generated. Since $\mathcal{V}_i \cong \mathcal{W}_{2i}^M \oplus \mathcal{W}_{2i+1}^M$, this will establish our theorem. Because the homology of configuration spaces of manifolds with finite type homology is finitely generated (as abelian groups), it suffices to show that these $\bigwedge(\text{Sym}^2 R)$ -modules have finite generation degree.

Before we proceed, we make some preliminary observations. By their definition, both \mathcal{U}_i and \mathcal{V}_i decompose into a direct sum of two $\bigwedge(\text{Sym}^2 R)$ -modules supported on sets of a given parity. Moreover, when k is even, $\mathcal{U}_i(k) = \mathcal{V}_i(k)$. When k is odd, $\mathcal{U}_i(k) = \mathcal{V}_{i-1}(k)$.

We also observe that, by combining Proposition 2.54, Proposition 3.24 and considering the spectral sequence associated to a filtered chain complex, we conclude the following results.

(a) If \mathcal{U}_i is finitely generated for $i \leq m$, then for all j ,

$$H_j(B_*^i(k)) \cong 0 \quad \text{for all } i \leq m, \text{ and all } k \text{ sufficiently large (depending on } i, j, \text{ and } M).$$

(b) If \mathcal{V}_i is finitely generated for $i < m$, then for all j ,

$$H_j(A_*^i(k)) \cong 0 \quad \text{for all } i < m, \text{ and all } k \text{ sufficiently large (depending on } i, j, \text{ and } M).$$

The proof of Theorem 3.11 implies that \mathcal{U}_i and \mathcal{V}_i both vanish for i strictly negative. This will serve as the base case of the following two step induction argument. We first assume that \mathcal{U}_i is finitely generated for $i \leq m$ and \mathcal{V}_i for $i < m$, and then prove that \mathcal{V}_m is finitely generated. Since $\mathcal{U}_i(k) = \mathcal{V}_i(k)$ for even k , for this step it is enough to show that $H_0^{\text{FIM}^+}(\mathcal{V}_m)(k) \cong 0$ for k sufficiently large and odd. In the second step, we assume that \mathcal{V}_i and \mathcal{U}_i are finitely generated for $i \leq m$, and then prove that \mathcal{U}_{m+1} is finitely

generated. Since $\mathcal{U}_i(k) = \mathcal{V}_{i-1}(k)$ for odd k , for this step it is enough to show $H_0^{\text{FIM}^+}(\mathcal{U}_{m+1})(k) \cong 0$ for k sufficiently large and even. Note that for k even, $H_0^{\text{FIM}^+}(\mathcal{U}_{m+1})(k) \cong H_0^{\text{FIM}^+}(\mathcal{V}_{m+1})(k)$.

Assume for the purposes of induction that \mathcal{U}_i is finitely generated for $i \leq m$ and \mathcal{V}_i is finitely generated for $i < m$. By Proposition 3.25, our goal is to show $H_0(A_*^m(k)) \cong 0$ for odd k sufficiently large. Let $E_{p,q}^r$ denote the (p,q) entry on the r th page of the arc resolution spectral sequence. By definition, when k is odd,

$$A_0^m(k) \cong E_{-1, m+\frac{k+1}{2}}^2(k) \quad \text{and} \quad H_0(A_*^m(k)) \cong E_{-1, m+\frac{k+1}{2}}^3(k).$$

Since the connectivity of the arc resolution is $(k-1)$ by Proposition 3.8, we know that for large k ,

$$E_{-1, m+\frac{k+1}{2}}^\infty(k) \cong 0.$$

There are no differentials out of $E_{-1, m+\frac{k+1}{2}}^r(k)$ so to prove that $H_0(A_*^i(k))$ vanishes we just need to rule out the possibility of nonzero differentials d^r into this group for $r > 2$. The domains of such differentials are $E_{-1+r, m-r+1+\frac{k+1}{2}}^r(k)$ for $r > 2$. When $r \geq 2m+4$, the proof of Theorem 3.11 implies $E_{-1+r, m-r+1+\frac{k+1}{2}}^2(k) \cong 0$ which rules out nonzero differentials with codomain $E_{-1, m+\frac{k+1}{2}}^r(k)$ for $r \geq 2m+4$. The differentials are shown in the case $m=1, k=7$ in Figure 38.

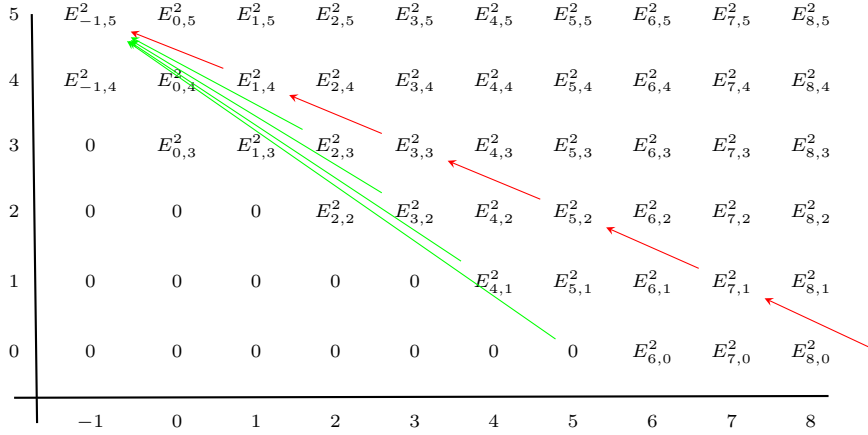


Figure 38: The complex $A_*^1(7)$ and the differentials d^3, d^4, d^5 , and d^6 (shown in green).

For $2 < r < 2m+4$, the groups $E_{-1+r, m-r+1+\frac{k+1}{2}}^3(k)$ are of the form $H_j(A_*^i(k))$ for $i < m$ and $0 \leq j \leq m+1$ or of the form $H_j(B_*^i(k))$ for $i \leq m$ and $0 \leq j \leq m+1$. Thus by observations (a) and (b), there is some uniform bound $K \in \mathbb{Z}$ such that these groups are all zero for $k > K$. In particular, for large k and $r > 2$, there cannot be nonzero differentials with codomain $E_{-1, m+\frac{k+1}{2}}^r(k)$. It follows that, for k sufficiently large and odd, that:

$$H_0(A_*^m(k)) \cong E_{-1, m+\frac{k+1}{2}}^3(k) \cong E_{-1, m+\frac{k+1}{2}}^\infty(k) \cong 0.$$

This establishes the first induction step.

Next we assume that \mathcal{V}_i and \mathcal{U}_i are finitely generated for $i \leq m$, and k is even. We wish to show $H_0(A_*^{m+1}(k)) \cong 0$ for even k sufficiently large. When k is even,

$$A_0^{m+1}(k) \cong E_{-1, m+1+\frac{k}{2}}^2(k) \quad \text{and} \quad H_0(A_*^{m+1}(k)) \cong E_{-1, m+1+\frac{k}{2}}^3(k).$$

Again, it suffices to show that $E_{-1, m+1+\frac{k}{2}}^3(k)$ is not the target of any nonzero differentials $d^r, r > 2$, once k is sufficiently large. The domains of the only possibly nonzero differentials are $E_{-1+r, m-r+2+\frac{k}{2}}^r(k)$ for $2 < r < 2m+5$. But for $2 < r < 2m+5$ the groups $E_{-1+r, m-r+2+\frac{k}{2}}^3(k)$ are one of $H_j(A_*^i(k))$ for $i \leq m$ and $0 \leq j \leq m+2$, or $H_j(B_*^i(k))$ for $i \leq m$ and $0 \leq j \leq m+1$. By observations (a) and (b) these groups vanish for large even k . This establishes the second induction step and the theorem. \square

3.5 Improved range in higher dimensions

Although Theorem 1.4 holds for manifolds of dimension $n \geq 3$, the result in higher dimensions is in a sense degenerate: the homology operation ψ is zero for $n \geq 3$, and the isomorphism of Corollary 1.5 is also the zero map. Thus, in high dimensions secondary representation stability manifests itself as an improved range for representation stability. We begin by showing that the arc resolution spectral sequence collapses at the E^2 -page if $\dim M > 2$. In this subsection, we work with integral coefficients.

Proposition 3.26. *If M is a noncompact connected manifold of dimension $n \geq 3$, the arc resolution spectral sequence collapses at the E^2 -page.*

Proof. Since $F_2(\mathbb{R}^n) \simeq S^{n-1}$, the class $\psi(1, 2) \in H_1(F_2(\mathbb{R}^n))$ is zero for $n \geq 3$. Since Browder operations are bilinear, the iterated products $\psi(1, \psi(2, \dots, \psi(k-1, k) \dots))$ vanishes in $H_{k-1}(F_k(\mathbb{R}^n))$. In particular, $t_{\psi(1, \psi(2, \dots, \psi(k-1, k) \dots))}$ is the zero map.

By Lemma 3.16, for $L = [1, \dots, [k-1, k], \dots] \in E_{k-1,0}^1(S)$, the differential $d^r(L)$ vanishes for $r < k$ and

$$d^k(L) = t_{\psi(1, \psi(\dots, \psi(k-1, k) \dots))}(y_0)$$

where y_0 is the class of a point in $H_0(F_0(M))$. Thus $d^k(L) = 0$ as well. For degree reasons, for $r > k$ the codomain of the differential d^r is zero, and $d^r(L) = 0$. From this calculation, the description of the E^2 -page given in Proposition 3.10, and the Leibniz rule (Lemma 3.18), we conclude that the differentials d^r vanishes for $r > 1$. \square

Using Proposition 3.26, we can prove an improved stable range for the homology of configuration spaces of higher-dimensional manifolds. This result was proven by Church, Ellenberg, and Farb for noncompact connected orientable manifolds [CEF15, Theorem 6.4.3], and we extend their result to all noncompact connected manifolds.

Theorem 3.27. *Let M be a noncompact connected manifold of dimension at least three. Then $\deg H_0^{\text{FI}}(H_i(F(M); \mathbb{Z})) \leq i$.*

Proof. Consider the spectral sequence described in Proposition 3.10. We proved in Proposition 3.8 that $E_{p,q}^\infty(S) \cong 0$ for $p + q \leq |S| - 2$. Proposition 3.26 implies that $E_{p,q}^\infty(S) \cong E_{p,q}^2(S)$ for all p and q . Since $H_0^{\text{FI}}(H_i(F(M)))_S \cong E_{-1,i}^2(S)$, the claim follows. \square

3.6 Conjectures and calculations

In this subsection, we make several conjectures. We give evidence for some of these conjectures by proving them in special cases.

Higher dimensional manifolds and higher order stability

We begin with some questions concerning configuration spaces. It seems likely that Theorem 1.4, on finite generation of the unstable homology groups $H_0^{\text{FI}}\left(H_{\frac{|S|+i}{2}}(F(M); R)\right)$ as a $\bigwedge(\text{Sym}^2 R)$ -module, can be strengthened and generalized. In particular, we have the following questions.

- Question 3.28.** (a) Is there a notion of tertiary and higher order representation stability that is present in the homology of configuration spaces?
- (b) What is the stable range for secondary representation stability?
- (c) Is there any form of nontrivial secondary representation stability for configuration spaces of higher dimensional manifolds?

We suggest a conjectural answer to all three questions. To state this conjecture, we need the following definition.

Definition 3.29. Define the following twisted (skew-)commutative algebras:

$$\mathfrak{L}_d^n = \begin{cases} \text{Sym } H_{d-1}(F_d(\mathbb{R}^n)), & (d-1)(n-1) \text{ even} \\ \bigwedge H_{d-1}(F_d(\mathbb{R}^n)), & (d-1)(n-1) \text{ odd.} \end{cases}$$

Note that $H_{d-1}(F_d(\mathbb{R}^n))$ is \mathcal{L}_d when $(d-1)(n-1)$ is odd. For $(d-1)(n-1)$ even, there is a similar description except with different signs. For $d=1$ and n arbitrary, \mathfrak{L}_d^n -modules are precisely FI-modules. For $d=2$, these \mathfrak{L}_d^n -modules are modules over $\bigwedge(\text{Sym}^2 R)$ if n is even and modules over $\text{Sym}(\bigwedge^2 R)$ if n is odd.

For a noncompact n -manifold M , the embedding $\mathbb{R}^n \sqcup M \hookrightarrow M$ induces maps

$$H_{d-1}(F_d(\mathbb{R}^n)) \otimes H_i(F_k(M)) \longrightarrow H_{i+(n-1)(d-1)}(F_{k+d}(M)).$$

For $d=1$, this gives the FI-module structure on $H_i(F(M))$ and for $d=n=2$, this gives the $\bigwedge(\text{Sym}^2 R)$ -module structure on \mathcal{W}_i^M . In general these embeddings induce \mathfrak{L}_d^n -module structures on the groups $\mathcal{W}[d]_i^M(S)$ defined as follows.

Definition 3.30. Let M be a noncompact connected manifold of dimension n and let

$$\mathcal{W}[d]_i^M(S) = H_0^{\mathfrak{L}_{d-1}^n} \left(\dots \left(H_0^{\mathfrak{L}_1^n} \left(H_{\frac{(n-1)(d-1)|S|+i}{d}}(F(M); R) \right) \right) \dots \right)_S$$

Note that we use the \mathfrak{L}_d^n -module structure on $\mathcal{W}[d]_i^M$ to define $\mathcal{W}[d+1]_i^M(S)$. For $d=1$, $\mathcal{W}[d]_i^M$ is just the FI-module $H_i(F(M))$. For $d=n=2$, $\mathcal{W}[d]_i^M$ is the $\bigwedge(\text{Sym}^2 R)$ -module \mathcal{W}_i^M . We conjecture that these modules have finite generation degree when M is sufficiently highly connected, and we conjecture an explicit stable range.

Conjecture 3.31. Let M be a noncompact manifold of dimension $n \geq 2$. If M is τ -connected with $\tau \geq \left\lfloor \frac{(n-1)(d-1)}{d} \right\rfloor$, then $H_0^{\mathfrak{L}_d^n}(\mathcal{W}[d]_i^M(S)) \cong 0$ for

$$|S| > \max \left(\frac{i(d^2 + d)}{n-1}, \frac{id}{\tau d - (n-1)(d-1)} \right).$$

The above conjecture can be interpreted as three separate conjectures, addressing the three parts of Question 3.28. Note that $\lfloor ((n-1)(d-1))/d \rfloor = 0$ when $n=2$ and thus for surfaces we are only assuming that the manifold is connected. Our conjecture for the stable range comes from our calculations in Proposition 3.32. Our heuristic for assuming that the manifold needs to be $\lfloor ((n-1)(d-1))/d \rfloor$ -connected is to bound the *slope* of certain homology classes, that is, the ratio of homological degree to the number of moving particles. This condition seems to ensure that the slope of all homology classes in the configuration space that “come from the topology of the manifold” is higher than those coming from $H_{d-1}(F_d(\mathbb{R}^n))$.

As support for Conjecture 3.31, we will next prove the result in the case that the manifold is \mathbb{R}^n . From now on, we work with integral coefficients.

Configurations of (punctured) Euclidean space

Cohen [CLM76, Chapter III] proved that the homology groups $H_*(F_k(\mathbb{R}^n))$ are the submodule of the free graded commutative algebra on the free graded lie algebra on the set $[k]$ such that in each product of brackets, every element of $[k]$ appears exactly once. The bracket is the E_n -Browder operation ψ^n and the product is \bullet (see Theorem 3.4). For example, a typical element of $H_{3(n-1)}(F_6(\mathbb{R}^n))$ is $2 \bullet \psi^n(1, 4) \bullet \psi^n(3, \psi^n(5, 6))$.

Proposition 3.32. Conjecture 3.31 is true for the integral homology of $M = \mathbb{R}^n$. Specifically, for $n > 1$,

$$H_0^{\mathfrak{L}_d^n}(\mathcal{W}[d]_i^{\mathbb{R}^n})(S) \cong 0 \quad \text{for all } |S| > \frac{i(d^2 + d)}{n-1}.$$

Proof. Cohen's description of $H_*(F_k(\mathbb{R}^n))$ allows us to compute the groups $\mathcal{W}[d]_i^{\mathbb{R}^n}$ explicitly. The \mathcal{L}_d^n -module structure on $\mathcal{W}[d]_i^{\mathbb{R}^n}$ is induced by stabilizing by $(d-1)$ nested E_n -Browder operations. Thus, the \mathcal{L}_d^n generators $H_0^{\mathcal{L}_d^n}(\mathcal{W}[d]_i^{\mathbb{R}^n})(S)$ are spanned by products of d or more nested Browder operations. The ratio of homological degree to number of particles for these classes is at least $\frac{(n-1)(d)}{d+1}$; see Theorem 3.4. The group $H_0^{\mathcal{L}_d^n}(\mathcal{W}[d]_i^{\mathbb{R}^n})(S)$ is defined as a subquotient (and in fact, for $M = \mathbb{R}^n$, is a submodule) of the homology group $H_{i+\frac{(n-1)(d-1)|S|}{d}}(F_S(\mathbb{R}^n))$. Thus, elements of $H_0^{\mathcal{L}_d^n}(\mathcal{W}[d]_i^{\mathbb{R}^n})(S)$ have a ratio of homological degree to number of particles given by $\frac{i}{|S|} + \frac{(n-1)(d-1)}{d}$. If

$$\frac{i}{|S|} + \frac{(n-1)(d-1)}{d} < \frac{(n-1)(d)}{d+1},$$

then the set of abelian group generators for $H_0^{\mathcal{L}_d^n}(\mathcal{W}[d]_i^{\mathbb{R}^n})(S)$ is empty and so $H_0^{\mathcal{L}_d^n}(\mathcal{W}[d]_i^{\mathbb{R}^n})(S)$ vanishes in the indicated range. \square

Remark 3.33. Conjecture 3.31 also holds when M is an open n -disk with punctures. Specifically, for such a manifold M with $n > 1$,

$$H_0^{\mathcal{L}_d^n}(\mathcal{W}[d]_i^M)(S) \cong 0 \quad \text{for all } |S| > \frac{i(d^2 + d)}{n-1}.$$

This remark can be proven using similar results of Cohen, but we do not include these details here.

Cohen's calculation completely determine the modules $\mathcal{W}_i^{\mathbb{R}^2}$. We describe the case of $i = 0$ in detail.

Proposition 3.34. *For the plane $M = \mathbb{R}^2$,*

$$\mathcal{W}_0^{\mathbb{R}^2} \cong M^{\text{FIM}^+}(0).$$

In particular, $\mathcal{W}_0^{\mathbb{R}^2}(2k) = H_0^{\text{FI}}(H_k(F(\mathbb{R}^2)))_{2k}$ is a rank- $\binom{(2k)!}{k!2^k}$ free module.

Proof. For any $k > 0$ the generators $H_0^{\text{FI}}(H_i(F(\mathbb{R}^2)))_k$ can be identified with a subgroup of the free abelian group $H_i(F_k(\mathbb{R}^2))$ where the products of iterated brackets have no degree-0 singleton factors. In particular, $H_0^{\text{FI}}(H_k(F(\mathbb{R}^2)))_{2k}$ has a basis indexed by the set of perfect matchings on $[2k]$, where the matching $\{a_i, b_i\}_{i=1}^k$ corresponds (up to sign) to the homology class $\psi(a_1, b_1) \bullet \psi(a_2, b_2) \bullet \cdots \bullet \psi(a_k, b_k)$, as in Figure 39. This description follows from the work of Cohen, or (after dualizing) the presentation

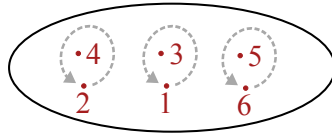


Figure 39: The basis element for $H_0^{\text{FI}}(H_3(F(\mathbb{R}^2)))_6$ corresponding the matching $\{\{4, 2\}, \{3, 1\}, \{5, 6\}\}$.

for $H^i(F_k(\mathbb{R}^2))$ computed by Arnold [Arn69, Theorem 1 and Corollary 3]. As an \mathfrak{S}_{2k} -representation, the group $H_0^{\text{FI}}(H_k(F(\mathbb{R}^2)))_{2k}$ is precisely $M^{\text{FIM}^+}(0)_{2k}$. Since $t_{\psi(a,b)}$ is the operation $x \mapsto \psi(a, b) \bullet x$, this identification is compatible with the FIM^+ structure. \square

The decomposition of the \mathfrak{S}_{2k} -representation $H_0^{\text{FI}}(H_k(F(\mathbb{R}^2); \mathbb{Q}))_{2k}$ into irreducible constituents of is given explicitly in Proposition 3.40.

Remark 3.35. These methods used to prove Proposition 3.34 can be used to perform other calculations. For example, if M is a punctured 2-disk, then $\mathcal{W}_0^M \cong M^{\text{FIM}^+}(0)$ and $\mathcal{W}_1^M \cong M^{\text{FIM}^+}(H_1(M)) \oplus M^{\text{FIM}^+}(\mathcal{L}_3)$.

Remark 3.36. The ideas of Proposition 3.32 can be used to prove other secondary representation stability results in situations where complete calculations exist. For example, it can be used to prove a version of secondary representation stability for the Cartesian powers of a connected space. Let (X, x_0) be a based space. For simplicity, we will take homology with coefficients in a field. Let X° be the FB-space whose value on a set S is X^S . The choice of the base-point gives X° the structure of an FI-space. Using the Künneth isomorphism, we see that if X is connected, then $\deg H_0^{\text{FI}}(H_i(X^\circ)) \leq i$. The functor

$$S \mapsto H_0^{\text{FI}}(H_{i+|S|}(X^\circ))_S$$

is naturally a module over the skew-tca $\bigwedge H_1(X)$. Again using the Künneth isomorphism, it is straightforward to verify that this module has generation degree at most i .

Computing \mathcal{W}_i^M for some surfaces M and small i

We now compute \mathcal{W}_i^M for some other surfaces and small i . Even for $i = 0$, the situation is very different for higher genus or nonorientable surfaces.

Proposition 3.37. *Let M be a connected surface. If M is not orientable or of genus greater than zero, then*

$$\mathcal{W}_0^M(0) \cong \mathbb{Z} \quad \text{and} \quad \mathcal{W}_0^M(2i) \cong 0 \quad \text{for } i > 0.$$

Proof. By definition,

$$\mathcal{W}_0^M(0) = H_0^{\text{FI}}(H_0(F(M)))_0 = H_0(F_0(M)).$$

As claimed, this is isomorphic to \mathbb{Z} for any connected manifold M .

To prove the vanishing of $\mathcal{W}_0^M(2i) = H_0^{\text{FI}}(H_i(F(M)))_{2i}$ for $i > 0$ we will first show, by assembling known results, that the map

$$H_0^{\text{FI}}(H_1(F(\mathbb{R}^2)))_2 \longrightarrow H_0^{\text{FI}}(H_1(F(M)))_2$$

induced by embedding $\mathbb{R}^2 \hookrightarrow M$ is zero. We begin with the case that M is nonorientable. Let \mathcal{M} denote the Möbius strip. Since \mathcal{M} is open, $H_1(F(\mathcal{M}))$ is an $\text{FI}_\#$ -module. By Church, Ellenberg, and Farb [CEF15, Theorem 4.1.5] (here Theorem 2.16) it follows that

$$H_1(F_2(\mathcal{M})) \cong \bigoplus_{\ell=0}^2 \text{Ind}_{\mathfrak{S}_\ell \times \mathfrak{S}_{2-\ell}}^{\mathfrak{S}_2} H_0^{\text{FI}}(H_1(F(\mathcal{M})))_\ell \boxtimes \mathbb{Z},$$

with \mathbb{Z} the trivial $\mathfrak{S}_{2-\ell}$ -representation. Since $H_1(F_1(\mathcal{M})) \cong H_1(\mathcal{M}) \cong \mathbb{Z}$ and $H_1(F_0(\mathcal{M})) \cong 0$, the component of $H_1(F_2(\mathcal{M}))$ generated in degrees $\ell = 0, 1$ is isomorphic to

$$\text{Ind}_{\mathfrak{S}_1 \times \mathfrak{S}_1}^{\mathfrak{S}_2} H_0^{\text{FI}}(H_1(F(\mathcal{M})))_1 \boxtimes \mathbb{Z} \cong \text{Ind}_{\mathfrak{S}_1 \times \mathfrak{S}_1}^{\mathfrak{S}_2} \mathbb{Z} \cong \mathbb{Z}^2,$$

the canonical \mathfrak{S}_2 permutation representation. Wang [Wan02, Lemma 1.6] showed that $H^1(F_2(\mathcal{M})) \cong \mathbb{Z}^2$ and that $H^2(F_2(\mathcal{M})) \cong \mathbb{Z}$ and hence is torsion free. We deduce that $H_1(F_2(\mathcal{M})) \cong \mathbb{Z}^2$ consists entirely of its $\ell = 1$ component: the component $H_0^{\text{FI}}(H_1(F(\mathcal{M})))_2$ generated in degree $\ell = 2$ is zero. Hence the map

$$H_0^{\text{FI}}(H_1(F(\mathbb{R}^2)))_2 \longrightarrow H_0^{\text{FI}}(H_1(F(\mathcal{M})))_2$$

is zero. For a general noncompact nonorientable surface M , the map

$$H_0^{\text{FI}}(H_1(F(\mathbb{R}^2)))_2 \longrightarrow H_0^{\text{FI}}(H_1(F(M)))_2$$

factors through $H_0^{\text{FI}}(H_1(F(\mathcal{M})))_2$, and so is zero.

From a presentation of $\pi_1(F_k(M))$ for M a noncompact, orientable positive genus surface (for example, Bellingeri [Bel04, Theorem 6.1]) we see that the map $H_1(F_2(\mathbb{R}^2)) \rightarrow H_1(F_2(M))$ is zero even before passing to the quotient module of FI generators.

To complete the argument, we recall the arc resolution spectral sequence described in Section 3.2. Let $E_{p,q}^r(2i)$ denote page r of the arc resolution spectral sequence on $2i$ particles. By the proof of

Theorem 3.11, the domain of any differentials $d_{p,q}^r$ with codomain $E_{-1,i}^r(2i)$ are zero for $r > 2$. Since $E_{p,q}^r(2i) = 0$ for $p < -1$, there are no nontrivial differentials out of the group $E_{-1,i}^r$. High connectivity of the arc resolution (Proposition 3.8) implies that $E_{-1,i}^\infty(2i) \cong 0$ for $i > 0$. Thus the differentials

$$d^2 : E_{1,i-1}^r(2i) \longrightarrow E_{-1,i}^r(2i)$$

are surjective for $i > 0$. Equivalently, the maps

$$\mathrm{Ind}_{\mathfrak{S}_{2i-2} \times \mathfrak{S}_2}^{\mathfrak{S}_{2i}} \mathcal{W}_0^M(2i-2) \boxtimes H_1(F_2(\mathbb{R}^2)) \longrightarrow \mathcal{W}_0^M(2i)$$

surject for all $i > 0$. We have shown that the map $H_1(F_2(\mathbb{R}^2)) \longrightarrow H_1(F_2(M))$ is zero if M is not orientable or has positive genus, and so for $i = 1$,

$$\mathrm{Ind}_{\mathfrak{S}_0 \times \mathfrak{S}_2}^{\mathfrak{S}_2} \mathcal{W}_0^M(0) \boxtimes H_1(F_2(\mathbb{R}^2)) \longrightarrow \mathcal{W}_0^M(2)$$

is the zero map. Since it is also surjective, $\mathcal{W}_0^M(2) \cong 0$. The claim for higher i then follows inductively, using the fact that only the zero group can be the surjective image of the zero group. \square

In Proposition 3.37 we saw that for nonorientable or positive genus surfaces M ,

$$\mathcal{W}_0^M(2i) = H_0^{\mathrm{FI}}(H_i(F(M)))_{2i} = 0 \quad \text{for } i > 0,$$

and this gives the following small improvement on known stable ranges.

Corollary 3.38. *Let M be a connected noncompact manifold which is not a (possibly punctured) 2-disk, and let $i > 0$. Then $H_i(F(M); \mathbb{Z})$ is generated in degree $\leq 2i - 1$.*

Example 3.39. Let $M = T_1$ be the once punctured torus, and take $R = \mathbb{Q}$. By comparing configuration spaces of T_1 to that of a closed torus, and by using a computer program based on work of Totaro [Tot96], John Wiltshire-Gordon [WG] calculated the following:

$$\dim_{\mathbb{Q}}(H_2(F_k(T_1); \mathbb{Q})) \cong 5 \binom{k}{2} + 2 \binom{k}{3},$$

so in particular:

$$\dim_{\mathbb{Q}}(H_0^{\mathrm{FI}}(H_2(F(T_1); \mathbb{Q}))_k) = \begin{cases} 5, & k = 2 \\ 2, & k = 3 \\ 0, & \text{otherwise.} \end{cases}$$

With Wiltshire-Gordon's calculation and the arc resolution spectral sequence, we are able to show that over \mathbb{Q} ,

$$\mathcal{W}_1^{T_1}(S) \cong \begin{cases} \mathbb{Q}^2, & |S| = 1, 3 \\ 0, & \text{otherwise.} \end{cases}$$

For $a \in S$, let m_a denote the class where one particle traverses the meridian in $H_1(F_{\{a\}}(M))$. This calculation also uses the fact that $t_{\psi(a,b)}(m_c) = t_{\psi(a,c)}(m_b)$. The symmetric group action on these vector spaces is trivial. Stabilizing by Browder operations induces an isomorphism between $\mathcal{W}_1^{T_1}(1)$ and $\mathcal{W}_1^{T_1}(3)$.

It would be interesting to have more explicit calculations of the FIM^+ -modules \mathcal{W}_i^M in other homological degrees, and for other surfaces.

The combinatorics of FIM^+ -modules

There has been considerable recent success in characterizing the structure of finitely generated modules over the category FI and certain relatives, and these results suggest a number of questions about what “representation stability” should mean for modules over FIM^+ . In Proposition 3.41, we describe the decomposition of free FIM^+ -modules over \mathbb{Q} into irreducible symmetric group representations, using a

calculation of $M^{\text{FIM}^+}(0)$ stated in Proposition 3.40. In Question 3.43, we pose some questions about the structure of finitely generated rational FIM^+ -modules.

Let $\mathfrak{B}_k \cong \mathfrak{S}_2 \wr \mathfrak{S}_k \subseteq \mathfrak{S}_{2k}$ denote the signed permutation group on k letters, the Coxeter group in type B_k/C_k . Let $V_{(1^k, \emptyset)}$ denote the 1-dimensional rational \mathfrak{B}_k -representation pulled back from the sign \mathfrak{S}_k -representation under the natural surjection $\mathfrak{B}_k \rightarrow \mathfrak{S}_k$. There are isomorphisms of \mathfrak{S}_{2k} -representations:

$$M^{\text{FIM}^+}(0)_{2k} \cong \text{Ind}_{\mathfrak{B}_k}^{\mathfrak{S}_{2k}} V_{(1^k, \emptyset)}.$$

The decompositions of these induced representations are described explicitly by Stembridge [Ste06, Page 7], a result which he attributes to Littlewood. These decompositions are as follows.

Proposition 3.40 (Littlewood, Stembridge [Ste06, Page 7]). *There are isomorphisms of \mathfrak{S}_{2k} -representations:*

$$M^{\text{FIM}^+}(0)_{2k} \cong \bigoplus_{\lambda \in D_{2k}} V_\lambda.$$

Here V_λ the irreducible \mathfrak{S}_{2k} -representation associated to the partition λ . A partition $\lambda \vdash 2k$ is in D_{2k} if and only if it has the following symmetry: when the associated young diagram (in English notation) is cut into two along the staircase shown in Figure 40, then the resultant two skew subdiagrams are

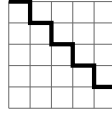


Figure 40: Staircase dividing young diagrams into two skew subdiagrams.

symmetric under reflection in the line of slope -1 .

Figure 41 illustrates this symmetry in the case $2k = 6$. Notably, identifying a partition in D_{2k} with

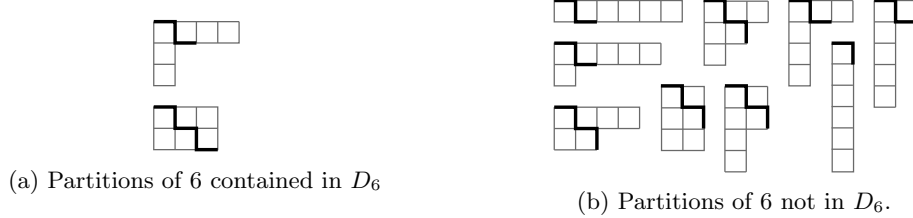


Figure 41: The set D_6 .

one of its skew subdiagrams puts D_{2k} in bijection with *strict* partitions of k , that is, partitions with distinct parts.

This computation of $M^{\text{FIM}^+}(0)$ allows us to use the Littlewood-Richardson rule to compute the decomposition of any free rational FIM^+ -module.

Proposition 3.41. *For integers $0 \leq d \leq k$, there are isomorphisms of rational \mathfrak{S}_k -representations*

$$M^{\text{FIM}^+}(d)_k \cong \begin{cases} \text{Ind}_{\mathfrak{S}_{k-d}}^{\mathfrak{S}_k} M^{\text{FIM}^+}(0)_{k-d}, & k \equiv d \pmod{2} \\ 0, & k \not\equiv d \pmod{2} \end{cases}$$

Given a rational \mathfrak{S}_d -representation W , the associated free FIM^+ -module

$$M^{\text{FIM}^+}(W) \cong M^{\text{FIM}^+}(d) \otimes_{\mathbb{Q}[\mathfrak{S}_d]} W$$

has the following decomposition:

$$M^{\text{FIM}^+}(W)_k \cong \begin{cases} \text{Ind}_{\mathfrak{S}_d \boxtimes \mathfrak{S}_{k-d}}^{\mathfrak{S}_k} W \boxtimes M^{\text{FIM}^+}(0)_{k-d}, & k \equiv d \pmod{2} \\ 0, & k \not\equiv d \pmod{2} \end{cases}$$

Example 3.42. For example, the first five nonzero components of the rational module $M^{\text{FIM}^+}(1)$ decompose as follows:

$$\begin{aligned}
M^{\text{FIM}^+}(1)_1 &\cong V_{\square} & M^{\text{FIM}^+}(1)_3 &\cong V_{\begin{smallmatrix} \square & \square \\ \square \end{smallmatrix}} & M^{\text{FIM}^+}(1)_5 &\cong V_{\begin{smallmatrix} \square & \square & \square \\ \square \end{smallmatrix}} \oplus V_{\begin{smallmatrix} \square & \square \\ \square & \square \end{smallmatrix}} \oplus V_{\begin{smallmatrix} \square & \square \\ \square \end{smallmatrix}} \\
M^{\text{FIM}^+}(1)_7 &\cong V_{\begin{smallmatrix} \square & \square & \square & \square \\ \square \end{smallmatrix}} \oplus V_{\begin{smallmatrix} \square & \square & \square \\ \square & \square \end{smallmatrix}} \oplus V_{\begin{smallmatrix} \square & \square & \square \\ \square \end{smallmatrix}} \oplus V_{\begin{smallmatrix} \square & \square & \square \\ \square & \square \end{smallmatrix}} \oplus V_{\begin{smallmatrix} \square & \square & \square \\ \square \end{smallmatrix}} \\
M^{\text{FIM}^+}(1)_9 &\cong V_{\begin{smallmatrix} \square & \square & \square & \square & \square \\ \square \end{smallmatrix}} \oplus V_{\begin{smallmatrix} \square & \square & \square & \square \\ \square & \square \end{smallmatrix}} \oplus V_{\begin{smallmatrix} \square & \square & \square & \square \\ \square \end{smallmatrix}} \oplus V_{\begin{smallmatrix} \square & \square & \square & \square \\ \square & \square \end{smallmatrix}} \oplus V_{\begin{smallmatrix} \square & \square & \square & \square \\ \square \end{smallmatrix}} \oplus V_{\begin{smallmatrix} \square & \square & \square & \square \\ \square & \square \end{smallmatrix}} \oplus V_{\begin{smallmatrix} \square & \square & \square & \square \\ \square \end{smallmatrix}}
\end{aligned}$$

In analogy to the other categories and (skew-)tca's that have been studied under the scope of “representation stability,” we pose the following questions. In essence: how should we understand “representation stability” for the category of FIM^+ -modules?

Question 3.43. What constraints does finite generation put on the irreducible representations appearing in a rational FIM^+ -module? Can the description of the irreducible constituents of $M^{\text{FIM}^+}(d)_k$ given in Proposition 3.41 be made more concrete? Are there parallels to the results for FI-modules due to Church–Ellenberg–Farb [CEF15, Theorem 1.13] and for FI_d -modules due to Ramos [Ram16, Theorem A]? Given a finitely generated rational FIM^+ -module \mathcal{V} , is there some operation on Young diagrams for constructing \mathcal{V}_{k+1} from \mathcal{V}_k in the spirit of Church and Farb’s *multiplicity stability* [CF13, Definition 1.1]? Does \mathcal{V}_k even determine \mathcal{V}_{k+1} for sufficiently large k ?

Stembridge [Ste06, Page 7] also gives explicit descriptions of the decompositions of the degree-0 free modules over $\text{Sym}(\text{Sym}^2\mathbb{Q})$, $\text{Sym}(\bigwedge^2\mathbb{Q})$, and $\bigwedge\bigwedge^2\mathbb{Q}$. Using the analogs of the formulas in Proposition 3.41, we can therefore compute the decompositions of all free modules over these categories. Though $\bigwedge(\text{Sym}^2\mathbb{Q})$ is most relevant to this paper, it would be very interesting to see a solution to Question 3.43 for any of these categories of modules.

Algebraic finiteness properties for twisted (skew-)commutative algebras

Conjecture 3.31 suggests the following purely algebraic questions.

Question 3.44. If \mathcal{W} is a finitely generated \mathfrak{L}_d^n -module and \mathcal{V} is a \mathfrak{L}_d^n -submodule, is \mathcal{V} necessarily finitely generated?

The main theorem of Church, Ellenberg, Farb and Nagpal from [CEFN14] shows that the answer is yes for $d = 1$. For R a field of characteristic zero, the answer is yes for $d = 2$. This is due to Nagpal, Sam and Snowden who address the case when n is odd [NSS16a, Theorem 1.1] and the case when n is even [NSS16b, Theorem 1.1]. The following question generalizes Church and Ellenberg’s results in [CE15] to the case of $d > 1$. For a (skew-)tca \mathcal{A} , let $H_i^{\mathcal{A}}$ denote the i th left derived functor of $H_0^{\mathcal{A}}$.

Question 3.45. Is there a function $f : \mathbb{N}_0 \times \mathbb{N}_0 \times \mathbb{N}_0 \rightarrow \mathbb{N}_0$ such that for all \mathfrak{L}_d^n -modules W with $\deg H_0^{\mathfrak{L}_d^n}(W) = g$ and $\deg H_1^{\mathfrak{L}_d^n}(W) = r$, then $\deg H_i^{\mathfrak{L}_d^n}(W) \leq f(g, r, i)$?

An affirmative answer to Question 3.45 for $d = n = 2$ would allow us to prove a quantitative version of Theorem 1.4. An affirmative answer to either of these two questions for $d > 2$ seems relevant to establishing tertiary and higher order representation stability results, although more ideas appear to be needed.

A Configuration spaces of compact manifolds

Until now, we assumed that the manifold M was not compact. In this section, we address the case of compact manifolds. Unfortunately, we do not know of any secondary stability phenomena for ordered configuration spaces of compact manifolds. However, we are able to improve known representation

stability results for ordered configuration spaces of compact manifolds in two ways: our results apply to both orientable and nonorientable manifolds, and they give an explicit stable range even with integral coefficients.

To do this, we use a result of Church and Ellenberg [CE15] on properties of the left derived functors of the functor H_0^{FI} . We then use a spectral sequence associated to the *puncture resolution* of configuration space $F_k(M)$ to prove our strengthened representation stability results. This is an adaptation of an argument of Randal-Williams [RW13] establishing rational homological stability for unordered configuration spaces.

A.1 Homology of FI-modules

Definition A.1. Let H_p^{FI} denote the p th left derived functor of H_0^{FI} . For an FI-module \mathcal{V} , call $\deg H_0^{\text{FI}}(\mathcal{V})$ the *generation degree* of \mathcal{V} and call $\deg H_1^{\text{FI}}(\mathcal{V})$ the *relation degree* of \mathcal{V} .

Church and Ellenberg [CE15] proved that the generation and relation degree of an FI-module bound the degrees of higher homology groups.

Theorem A.2. (Church–Ellenberg [CE15, Theorem A]) *Let \mathcal{V} be an FI-module with generation degree $d_{\mathcal{V}}$ and relation degree $r_{\mathcal{V}}$. Then*

$$\deg H_p(\mathcal{V}) - p \leq d_{\mathcal{V}} + r_{\mathcal{V}} - 1.$$

This theorem implies the following.

Proposition A.3. *Let*

$$\mathcal{U} \xrightarrow{\varphi} \mathcal{V} \xrightarrow{\rho} \mathcal{W}$$

be a sequence of FI-modules with $\rho \circ \varphi = 0$. Assume that $\mathcal{U}, \mathcal{V}, \mathcal{W}$ have generation degree $d_{\mathcal{U}}, d_{\mathcal{V}}$, and $d_{\mathcal{W}}$ and relation degree $r_{\mathcal{U}}, r_{\mathcal{V}}$, and $r_{\mathcal{W}}$ respectively. Then the FI-module $\frac{\ker \rho}{\text{im } \varphi}$ has generation degree

$$\deg H_0^{\text{FI}}\left(\frac{\ker \rho}{\text{im } \varphi}\right) \leq \max\left(r_{\mathcal{V}}, 1 + d_{\mathcal{W}} + \max(r_{\mathcal{W}}, d_{\mathcal{V}})\right)$$

and relation degree

$$\deg H_1^{\text{FI}}\left(\frac{\ker \rho}{\text{im } \varphi}\right) \leq \max\left(d_{\mathcal{U}}, r_{\mathcal{V}}, 2 + d_{\mathcal{W}} + d_{\mathcal{V}} + \max(r_{\mathcal{W}}, d_{\mathcal{V}})\right).$$

We remark that if \mathcal{U}, \mathcal{V} are \mathcal{W} are $\text{FI}_{\#}$ -modules, then they have relation degree zero [CE15, Corollary 3.4], and hence the bounds $r_{\mathcal{U}}, r_{\mathcal{V}}$, and $r_{\mathcal{W}}$ may be taken to be zero. If we additionally assume that the maps φ and ρ are maps of $\text{FI}_{\#}$ -modules, the result is greatly simplified: generation degree of $\text{FI}_{\#}$ -modules does not increase upon passing to subquotients [CEF15, Theorem 4.1.5]. In our application, Theorem A.12, some of the FI-modules have an $\text{FI}_{\#}$ -module structure, however, the maps may not respect this structure.

Proof of Proposition A.3. First observe the bounds obtained from the following surjective maps:

$$\begin{aligned} \mathcal{U} &\xrightarrow{\varphi} \text{im } \varphi && \text{so } \deg \text{im } \varphi \leq d_{\mathcal{U}} \\ \mathcal{W} &\twoheadrightarrow \frac{\mathcal{W}}{\text{im } \rho} = \text{coker } \rho && \text{so } \deg \text{coker } \rho \leq d_{\mathcal{W}} \\ \mathcal{V} &\xrightarrow{\rho} \text{im } \rho && \text{so } \deg \text{im } \rho \leq d_{\mathcal{V}} \end{aligned}$$

Now consider the short exact sequence:

$$0 \longrightarrow \text{im } \rho \hookrightarrow \mathcal{W} \twoheadrightarrow \frac{\mathcal{W}}{\text{im } \rho} \longrightarrow 0$$

and the associated long exact sequence in FI-homology.

$$\begin{array}{ccccccccccc}
\longrightarrow & H_2^{\text{FI}}(\text{im } \rho) & \longrightarrow & H_2^{\text{FI}}(\mathcal{W}) & \longrightarrow & H_2^{\text{FI}}(\mathcal{W}/\text{im } \rho) & \longrightarrow & H_1^{\text{FI}}(\text{im } \rho) & \longrightarrow & H_1^{\text{FI}}(\mathcal{W}) & \longrightarrow & H_1^{\text{FI}}(\mathcal{W}/\text{im } \rho) & \longrightarrow & H_0^{\text{FI}}(\text{im } \rho) & \longrightarrow & H_0^{\text{FI}}(\mathcal{W}) & \longrightarrow \\
\text{deg:} & (5) & & (3) & & (4) & & r_{\mathcal{W}} & & (2) & & \leq d_{\mathcal{V}} & & d_{\mathcal{W}}
\end{array}$$

The bounds $\deg H_1^{\text{FI}}(\mathcal{W}) = r_{\mathcal{W}}$ and $\deg H_0^{\text{FI}}(\text{im } \rho) \leq d_{\mathcal{V}}$ imply

$$\deg H_1^{\text{FI}}(\mathcal{W}/\text{im } \rho) \leq \max(r_{\mathcal{W}}, d_{\mathcal{V}}). \quad (2)$$

Since $\deg H_0^{\text{FI}}(\mathcal{W}/\text{im } \rho) \leq d_{\mathcal{W}}$ and $H_1^{\text{FI}}(\mathcal{W}/\text{im } \rho) \leq \max(r_{\mathcal{W}}, d_{\mathcal{V}})$ by Inequality (2), Theorem A.2 implies that

$$\deg H_2^{\text{FI}}(\mathcal{W}/\text{im } \rho) \leq 1 + d_{\mathcal{W}} + \max(r_{\mathcal{W}}, d_{\mathcal{V}}). \quad (3)$$

Again using the long exact sequence, the bounds $H_2^{\text{FI}}(\mathcal{W}/\text{im } \rho) \leq 1 + d_{\mathcal{W}} + \max(r_{\mathcal{W}}, d_{\mathcal{V}})$ from Inequality (3) and the bound $\deg H_1^{\text{FI}}(\mathcal{W}) = r_{\mathcal{W}}$ imply

$$\begin{aligned}
\deg H_1^{\text{FI}}(\text{im } \rho) &\leq \max(r_{\mathcal{W}}, 1 + d_{\mathcal{W}} + \max(r_{\mathcal{W}}, d_{\mathcal{V}})) \\
&= 1 + d_{\mathcal{W}} + \max(r_{\mathcal{W}}, d_{\mathcal{V}})
\end{aligned} \quad (4)$$

Since $\deg H_0^{\text{FI}}(\text{im } \rho) \leq d_{\mathcal{V}}$ and $\deg H_1^{\text{FI}}(\text{im } \rho) \leq 1 + d_{\mathcal{W}} + \max(r_{\mathcal{W}}, d_{\mathcal{V}})$ by Inequality (4), by Theorem A.2

$$\deg H_2^{\text{FI}}(\text{im } \rho) \leq 2 + d_{\mathcal{W}} + d_{\mathcal{V}} + \max(r_{\mathcal{W}}, d_{\mathcal{V}}). \quad (5)$$

Next consider the short exact sequence:

$$0 \longrightarrow \ker \rho \hookrightarrow \mathcal{V} \xrightarrow{\rho} \text{im } \rho \longrightarrow 0$$

and the associated long exact sequence in FI-homology.

$$\begin{array}{ccccccccccc}
\longrightarrow & H_2^{\text{FI}}(\text{im } \rho) & \longrightarrow & H_1^{\text{FI}}(\ker \rho) & \longrightarrow & H_1^{\text{FI}}(\mathcal{V}) & \longrightarrow & H_1^{\text{FI}}(\text{im } \rho) & \longrightarrow & H_0^{\text{FI}}(\ker \rho) & \longrightarrow & H_0^{\text{FI}}(\mathcal{V}) & \longrightarrow & H_0^{\text{FI}}(\text{im } \rho) & \longrightarrow & 0 \\
\text{deg:} & (5) & & (7) & & r_{\mathcal{V}} & & (4) & & (6) & & d_{\mathcal{V}} & & \leq d_{\mathcal{V}}
\end{array}$$

The bounds $\deg H_1^{\text{FI}}(\mathcal{V}) = d_{\mathcal{V}}$ and $\deg H_1^{\text{FI}}(\text{im } \rho) \leq 1 + d_{\mathcal{W}} + \max(r_{\mathcal{W}}, d_{\mathcal{V}})$ by Inequality (4) imply

$$\deg H_0^{\text{FI}}(\ker \rho) \leq \max(r_{\mathcal{V}}, 1 + d_{\mathcal{W}} + \max(r_{\mathcal{W}}, d_{\mathcal{V}})). \quad (6)$$

Since $\deg H_1^{\text{FI}}(\mathcal{V}) = r_{\mathcal{V}}$ and $\deg H_2^{\text{FI}}(\text{im } \rho) \leq 2 + d_{\mathcal{W}} + d_{\mathcal{V}} + \max(r_{\mathcal{W}}, d_{\mathcal{V}})$ by Inequality (5) we see

$$\deg H_1^{\text{FI}}(\ker \rho) \leq \max(r_{\mathcal{V}}, 2 + d_{\mathcal{W}} + d_{\mathcal{V}} + \max(r_{\mathcal{W}}, d_{\mathcal{V}})). \quad (7)$$

Finally, consider the short exact sequence:

$$0 \longrightarrow \text{im } \varphi \hookrightarrow \ker \rho \longrightarrow \frac{\ker \rho}{\text{im } \varphi} \longrightarrow 0$$

and the associated long exact sequence in FI-homology.

$$\begin{array}{ccccccc}
\longrightarrow & H_1^{\text{FI}}(\ker \rho) & \longrightarrow & H_1^{\text{FI}}\left(\frac{\ker \rho}{\text{im } \varphi}\right) & \longrightarrow & H_0^{\text{FI}}(\text{im } \varphi) & \longrightarrow & H_0^{\text{FI}}(\ker \rho) & \longrightarrow & H_0^{\text{FI}}\left(\frac{\ker \rho}{\text{im } \varphi}\right) & \longrightarrow & 0 \\
\text{deg:} & (7) & & (9) & & \leq d_{\mathcal{U}} & & (6) & & (8)
\end{array}$$

Inequality (6) states $\deg H_0^{\text{FI}}(\ker \rho) \leq \max(r_{\mathcal{V}}, 1 + d_{\mathcal{W}} + \max(r_{\mathcal{W}}, d_{\mathcal{V}}))$ and so

$$\deg H_0^{\text{FI}}\left(\frac{\ker \rho}{\text{im } \varphi}\right) \leq \max\left(r_{\mathcal{V}}, 1 + d_{\mathcal{W}} + \max(r_{\mathcal{W}}, d_{\mathcal{V}})\right). \quad (8)$$

By Inequality (7) $\deg H_1^{\text{FI}}(\ker \rho) \leq \max(r_{\mathcal{V}}, 2 + d_{\mathcal{W}} + d_{\mathcal{V}} + \max(r_{\mathcal{W}}, d_{\mathcal{V}}))$ and the bound $\deg H_0^{\text{FI}}(\text{im } \varphi) \leq d_{\mathcal{U}}$ we have

$$\deg H_1^{\text{FI}}\left(\frac{\ker \rho}{\text{im } \varphi}\right) \leq \max\left(d_{\mathcal{U}}, r_{\mathcal{V}}, 2 + d_{\mathcal{W}} + d_{\mathcal{V}} + \max(r_{\mathcal{W}}, d_{\mathcal{V}})\right). \quad (9)$$

This completes the proof. \square

Proposition A.3 specializes to the following result on spectral sequences of FI-modules.

Corollary A.4. *Consider a (homologically graded) spectral sequence $E_{p,q}^r$ of FI-modules with differentials:*

$$d^r : E_{p,q}^r \rightarrow E_{p-r,q+r-1}^r.$$

If $E_{p+r,q-r+1}^r, E_{p,q}^r$, and $E_{p-r,q+r-1}^r$ have generation degrees $D_{p+r,q-r+1}^r, D_{p,q}^r, D_{p-r,q+r-1}^r$ and relation degrees $R_{p+r,q-r+1}^r, R_{p,q}^r, R_{p-r,q+r-1}^r$, respectively, then $E_{p,q}^{r+1}$ has generation degree

$$\deg H_0^{\text{FI}}(E_{p,q}^{r+1}) \leq \max\left(R_{p,q}^r, 1 + D_{p-r,q+r-1}^r + \max(R_{p-r,q+r-1}^r, D_{p,q}^r)\right)$$

and relation degree

$$\deg H_1^{\text{FI}}(E_{p,q}^{r+1}) \leq \max\left(D_{p+r,q-r+1}^r, R_{p,q}^r, 2 + D_{p-r,q+r-1}^r + D_{p,q}^r + \max(R_{p-r,q+r-1}^r, D_{p,q}^r)\right).$$

Consider a (cohomologically graded) spectral sequence $\tilde{E}_r^{p,q}$ of FI-modules with differentials:

$$d_r : \tilde{E}_r^{p,q} \rightarrow \tilde{E}_r^{p+r,q-r+1}.$$

If $\tilde{E}_r^{p-r,q+r-1}, \tilde{E}_r^{p,q}$, and $\tilde{E}_r^{p+r,q-r+1}$ have generation degrees $D_r^{p-r,q+r-1}, D_r^{p,q}, D_r^{p+r,q-r+1}$ and relation degrees $R_r^{p-r,q+r-1}, R_r^{p,q}, R_r^{p+r,q-r+1}$, respectively, then $\tilde{E}_{r+1}^{p,q}$ has generation degree

$$\deg H_0^{\text{FI}}(\tilde{E}_{r+1}^{p,q}) \leq \max\left(R_r^{p,q}, 1 + D_r^{p+r,q-r+1} + \max(R_r^{p+r,q-r+1}, D_r^{p,q})\right)$$

and relation degree

$$\deg H_1^{\text{FI}}(\tilde{E}_{r+1}^{p,q}) \leq \max\left(D_r^{p-r,q+r-1}, R_r^{p,q}, 2 + D_r^{p+r,q-r+1} + D_r^{p,q} + \max(R_r^{p+r,q-r+1}, D_r^{p,q})\right).$$

A.2 The puncture resolution and representation stability

In this subsection, we prove a quantitative integral representation stability result for the cohomology of the ordered configuration spaces of a manifold M , where M may be compact and need not be orientable. We first consider the cohomology configuration spaces of noncompact manifolds. We then use these results to study the cohomology of configuration spaces of closed manifolds, using a semi-simplicial space called the puncture resolution. In this section, we work with integral coefficients for homology and cohomology.

Recall that the cohomology of the ordered configuration spaces of particles in arbitrary manifolds has a natural FI-module structure, which comes from a co-FI-space structure on the configuration spaces (see Section 3.1).

Definition A.5. Given a finitely generated abelian group A , let $\gamma(A)$ denote the minimum number of generators of A .

Proposition A.6. *Let M be a finite type noncompact connected manifold of dimension at least two. There exists polynomials $P_{M,i}$ and $P^{M,i}$ of degrees at most $2i$ such that*

$$\gamma(H_i(F_k(M); \mathbb{Z})) \leq P_{M,i}(k) \quad \text{and} \quad \gamma(H^i(F_k(M); \mathbb{Z})) \leq P^{M,i}(k).$$

Proof. The proof of Church–Ellenberg–Farb [CEF15, Theorem 4.1.7] shows that, for an FI \sharp -module \mathcal{V} with each \mathcal{V}_k finitely generated as an abelian group, then \mathcal{V} is generated in degree $\leq N$ if and only if $\gamma(\mathcal{V}_k)$ is bounded by a degree N polynomial in k . Combining this result with Theorem 3.11 implies the polynomial bound for homology. The polynomial bound for cohomology follows from the polynomial bound for homology and the universal coefficient theorem. \square

Using the proof of [CEF15, Theorem 4.1.7] in the opposite direction, we deduce the following cohomological analogue of Theorem 3.11.

Corollary A.7. *Let M be a noncompact connected manifold of dimension at least two with finite type homology. Then $\deg H_0^{\text{FI}}(H^i(F(M); \mathbb{Z})) \leq 2i$.*

Remark A.8. We can prove the polynomial bounds of Proposition A.6 directly by considering the fibration $F_k(M) \rightarrow F_{k-1}(M)$. With this approach it is possible to obtain improved stability bounds for higher dimensional manifolds with homology that vanishes in a range; compare to results of Church [Chu12, Theorem 4.1].

We now recall the *puncture resolution* from Randal-Williams [RW13], which will allow us to leverage our result for noncompact manifolds to prove a result for compact manifolds.

Definition A.9. Fix a positive integer k . We define the *puncture resolution* of the configuration space $F_k(M)$ as follows. Let:

$$\text{Pun}_p(F_k(M)) = \bigsqcup_{(x_0, \dots, x_p) \in F_{\{0, \dots, p\}}(M)} F_k(M - \{x_0, \dots, x_p\}).$$

Let $\text{Pun}_p(C_k(M))$ denote the quotient of $\text{Pun}_p(F_k(M))$ by the natural S_k action.

See Figure 42 for an illustration. For fixed k , the spaces $\text{Pun}_j(F_k(M))$ assemble to form an

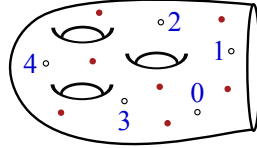


Figure 42: An element of $\text{Pun}_4(C_7(M))$.

augmented semi-simplicial space. The i th face map is induced by the inclusion of $M - \{x_0, \dots, x_p\}$ into $M - \{x_0, \dots, \hat{x}_i, \dots, x_p\}$. There is a homeomorphism $\text{Pun}_{-1}(F_k(M)) \cong F_k(M)$. Randal-Williams proved the following [RW13, Section 9.4].

Theorem A.10 (Randal-Williams). *Let M be a manifold. The map $||\text{Pun}_\bullet(C_k(M))|| \rightarrow C_k(M)$ is a weak equivalence.*

A proof similar to that of Proposition 3.8 gives the following.

Proposition A.11. *Let M be a manifold. The map $||\text{Pun}_\bullet(F_k(M))|| \rightarrow F_k(M)$ is a weak equivalence.*

For fixed p , the spaces $\text{Pun}_p(F_k(M))$ form a co-FI-space with respect to the usual particle deletion maps. This commutes with the semi-simplicial structure, and so $\text{Pun}_\bullet(F(M))$ has the structure of an augmented semi-simplicial co-FI-space. Since punctured manifolds are noncompact, for fixed $p \geq 0$, the p -simplices $\text{Pun}_p(F(M))$ form a homotopy FI \sharp -space. Note that this FI \sharp -structure does not respect the semi-simplicial structure.

Theorem A.12. *Let M be a finite type connected manifold of dimension at least 2. Then the FI-module $H^p(F(M))$ has generation degree at most $21(p+1)(1+\sqrt{2})^{p-2}$ and relation degree at most $28(p+1)(1+\sqrt{2})^{p-2}$.*

Proof. Consider the cohomological geometric realization spectral sequence associated to $\text{Pun}_\bullet(F(M))$, viewed as a *nonaugmented* co-FI semi-simplicial space. This gives a spectral sequence of FI-modules with

$$E_1^{p,q}(k) = H^p(\text{Pun}_q(F_k(M))) \quad \text{for } p, q \geq 0$$

which converges to

$$H^{p+q}(|\text{Pun}_\bullet(F_k(M))|) \cong H^{p+q}(F_k(M)).$$

Since, as spaces, the simplices $\text{Pun}_q(F_k(M))$ are a disjoint union of k -particle configuration spaces of the manifold M with $(q+1)$ punctures, it follows from Corollary A.7 and the FI \sharp -structure that

$$\deg H_0^{\text{FI}}(E_1^{p,q}) \leq 2p \quad \text{and} \quad H_i^{\text{FI}}(E_1^{p,q}) \cong 0 \quad \text{for } i > 0.$$

For computational simplicity (in the interest of producing a *linear* recurrence) we will use the weaker bounds on generation degree and relation degree at $E_1^{p,q}$

$$D_1^{p,q} \leq 2(p+q) \quad R^{p,q} \leq 0.$$

By Corollary A.4, the FI-module $E_2^{p,q}$ then has generation degree at most $(4(p+q)+3)$ and relation degree at most $(6(p+q)+4)$. For $r > 1$, the relation degree of $E_r^{p+q, q-r+1}$ becomes the largest term appearing in the formulas in Corollary A.4 and determines the maxima. In general, for $r > 1$, Corollary A.4 implies that if $E_r^{p,q}$ has generation degree at most $(a(p+q)+b)$ and relation degree at most $(c(p+q)+d)$, then $E_{r+1}^{p,q}$ will have generation degree at most $(a+c)(p+q) + (a+c+b+d+1)$ and relation degree at most $(2a+c)(p+q) + (a+2b+c+d+2)$. In other words, the transition from the E_r to E_{r+1} is the map

$$\begin{bmatrix} a \\ b \\ c \\ d \\ 1 \end{bmatrix} \mapsto \begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 \\ 2 & 0 & 1 & 0 & 0 \\ 1 & 2 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \\ 1 \end{bmatrix}$$

Given our initial bounds on $E_2^{p,q}$, the bounds on $E_{r+1}^{p,q}$ are given by

$$\begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 \\ 2 & 0 & 1 & 0 & 0 \\ 1 & 2 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}^{r-1} \begin{bmatrix} 4 \\ 3 \\ 6 \\ 4 \\ 1 \end{bmatrix}$$

This matrix has characteristic polynomial $(x-1)(x^2-2x-1)^2$ and maximum eigenvalue $(1+\sqrt{2})$. It is not diagonalizable, but by decomposing this initial vector into generalized eigenvectors we see that this product is equal to:

$$\begin{aligned} & \begin{bmatrix} 0 \\ -1 \\ 0 \\ -1 \\ 1 \end{bmatrix} + \left((1-\sqrt{2})^{r-1} + \left(\frac{2}{5} - \frac{3}{10}\sqrt{2} \right) (r-1)(1-\sqrt{2})^{r-2} \right) \begin{bmatrix} 0 \\ 5 - \frac{15}{4}\sqrt{2} \\ 0 \\ \frac{15}{2} - 5\sqrt{2} \\ 0 \end{bmatrix} + (1-\sqrt{2})^{r-1} \begin{bmatrix} 2 - \frac{3}{2}\sqrt{2} \\ -3 + \frac{17}{8}\sqrt{2} \\ 3 - 2\sqrt{2} \\ -5 + \frac{7}{2}\sqrt{2} \\ 0 \end{bmatrix} \\ & + \left((1+\sqrt{2})^{r-1} + \left(\frac{2}{5} + \frac{3}{10}\sqrt{2} \right) (r-1)(1+\sqrt{2})^{r-2} \right) \begin{bmatrix} 0 \\ 5 + \frac{15}{4}\sqrt{2} \\ 0 \\ \frac{15}{2} + 5\sqrt{2} \\ 0 \end{bmatrix} + (1+\sqrt{2})^{r-1} \begin{bmatrix} 2 + \frac{3}{2}\sqrt{2} \\ -3 - \frac{17}{8}\sqrt{2} \\ 3 + 2\sqrt{2} \\ -5 - \frac{7}{2}\sqrt{2} \\ 0 \end{bmatrix} \end{aligned}$$

The components of this vector are bounded above by the components of the vector

$$\begin{bmatrix} 12(1 + \sqrt{2})^{r-2} \\ (12 + 9(r-1))(1 + \sqrt{2})^{r-2} \\ 15(1 + \sqrt{2})^{r-2} \\ (15 + 13(r-1))(1 + \sqrt{2})^{r-2} \\ 1 \end{bmatrix}.$$

Thus $E_{r+1}^{p,q}$ has generation degree at most $12(1 + \sqrt{2})^{r-2}(p+q) + (12 + 9(r-1))(1 + \sqrt{2})^{r-2}$ and relation degree at most $15(1 + \sqrt{2})^{r-2}(p+q) + (15 + 13(r-1))(1 + \sqrt{2})^{r-2}$. The spectral sequence has converged at (p, q) once $r \geq p + q + 1$. We conclude that $H^{p+q}(F_k(M))$ has generation degree at most

$$21(p + q + 1)(1 + \sqrt{2})^{p+q-2}$$

and relation degree at most

$$28(p + q + 1)(1 + \sqrt{2})^{p+q-2}.$$

□

Remark A.13. Using the theory of weight and stability from Church–Ellenberg–Farb [CEF15], we may obtain a version of Theorem A.12 with a linear stability range using rational coefficients. It seems likely that a linear range should also hold with integer coefficients; however, the techniques of this paper are not sufficient to establish an integral linear bound.

Remark A.14. Corollary A.4 has many other application. For example, it can be used to make the stability ranges in Kupers–Miller’s work [KM15b] explicit and apply to manifolds with arbitrary fundamental group. It can also be used to prove an integral version of Jiménez Rolland’s result [JR15, Theorem 6.6].

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